AN

EASY INTRODUCTION

TO THE HIGHER TREATISES ON THE

CONIC SECTIONS.

BY THE

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PREFACE.

The present Introduction has resulted not from theorizing on what may be expedient, but from long experience of what is needful. The author has, from time to time, had applications for assistance in the study of such treatises on the Conic Sections as those of Salmon, Hymers, Puckle, and Todhunter. He has accordingly ascertained the nature of the difficulties that perplex and discourage many readers of those treatises, and has endeavoured in this publication to provide a first course of lessons and exercises which, he thinks, cannot fail of qualifying the young student for reading with intelligence and profit the masterly works referred to. Todhunter's book is especially recommended as the most appropriate sequel to this Introduction.

It should be remarked that the name *Conic Sections* chiefly refers to the Ellipse, Hyperbola, and Parabola, as curves that can be obtained by means of a right cone variously intersected by a plane.

London: *October*, 1866.

AN

EASY INTRODUCTION

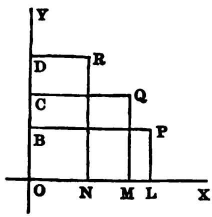
TO

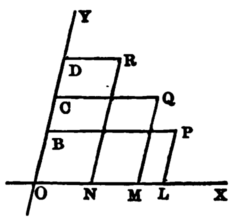
CONIC SECTIONS.

COORDINATES OF A POINT.

1. IN Plane Geometry the term *Ordinate* means one of any number of parallel lines drawn from successive points in one plane, to meet a straight line in the same plane perpendicularly or at any angle.

In the following diagrams, OX and OY are intersecting straight lines. The successive lines PL, QM, RN, YO are





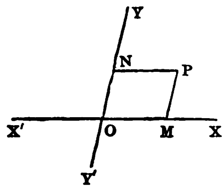
drawn *ordinately*, or in parallel succession, to meet OX; and the lines DR, CQ, BP, OX are drawn *ordinately* to OY. In one of the diagrams the parallels are perpendicular to the line they meet; in the other they are oblique.

2. The lines PL, PB, being both drawn from P, are

together called *Coordinates* of that point; in like manner, QM, QC are coordinates of the point Q; and so forth.

But the term *ordinate* is usually restricted to the lines that are parallel to OY; and instead of those parallel to OX we use the equivalent magnitudes OL, OM, ON, which, as being portions of OX *cut off* by the ordinates, are called *abscissæ*, and each of them an *abscissa*. Thus, QM is the ordinate and OM the abscissa of the point Q; and OM, QM are the coordinates of that point.

3. If, in one plane, two unlimited straight lines, X′X, YY′, be drawn intersecting, at any angle, at the fixed point O, the position of any other point in the plane may be referred to these lines, and expressed in terms supplied by them. For a point must be on the right or on the left of YY′, or else in the line YY′; and must also be above or below X′X, or else in the line X′X. Thus, the point P is situated on the right of Y′Y, and above X′X; that is, it is within the angle YOX; and if the lengths of PN and PM be known to be, respectively, 4 and 3, then, measuring OM=4 from O towards X, and ON=3 from O towards Y, we shall have indicated, by means of X′X and YY′, the position of P, viz. that P's position is at the intersection of lines drawn respectively parallel to OY and OX from the points M and N.



4. The lines YOY′ and X′OX are called the *Axes of Coordinates*; the former, by itself, the axis of ordinates, as being the line on which ordinates are measured; the latter, by itself, the axis of abscissæ, as being the line on which abscissæ are measured.

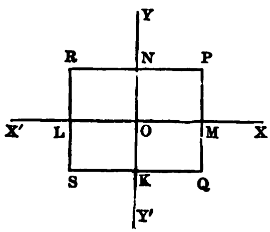
But, in general reasonings, an unknown or variable abscissa is denoted by the letter x, and its corresponding ordinate by the letter y; and therefore it is usual to call the axis of abscissæ the axis of x, and that of ordinates the axis of y.

The point O, where the coordinates intersect, is called the *Origin of Coordinates*, because we are to suppose the lines OX, OX′, OY, OY′ all to be drawn from O; OX to the right, OX′ to the left, OY upwards, OY′ downwards.

The coordinates are called *Rectangular* or *Oblique*, according as they do or do not intersect at right angles.

In the following part of this introductory treatise, we shall always suppose the axes to be *rectangular*.

5. The position, then, of P is known, if OM and PM are known. Suppose that the lengths of these coordinates are, respectively, a and b; we characterise the position of P thus, $x=a$, $y=b$, which are called equations to the point P.



For shortness, we often speak of the point (a, b), meaning the point which has a for the length of its abscissa, and b for that of its ordinate. So, the point (4, 3) means the point whose coordinates are $x=4$, $y=3$.

6. Suppose it is required to determine the geometrical position of the point $(-10, 7)$.

Here, since $x=-10$, and $y=7$, we take ON$=7$ *upward* from the origin, and OL$=10$ to the *left* of the origin; and, completing the parallelogram OR, we obtain R, the required position, which is within the angle YOX′. This procedure will be readily understood by the student who has read Chap. I. of the Author's *Treatise on Trigonometry*. The point O being the fixed origin of coordinates, an abscissa measured to the right of O, viz. along OX, is accounted positive; one measured to the left of O, viz. along OX′, is therefore negative; and, in like manner, an ordinate measured upwards from O, along OY, being accounted positive, one measured downwards from O, viz. along OY′, is to be accounted negative.

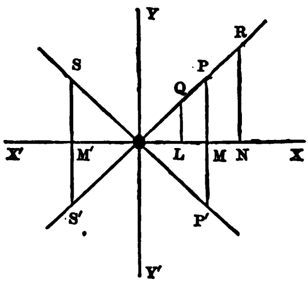
Suppose the absolute magnitude of OL$=$that of OM$=$10, and the absolute magnitude of OK$=$that of ON$=$7; then, according to the above conventional understanding, the point P is (10, 7), Q is (10, -7), R is (-10, 7), S is (-10, -7).

If a point whose abscissa is $=a$ be situated on the line OX, then the equations of the point are $x=a$, $y=0$, or the point is $(a, 0)$. And if a point whose ordinate is $=b$ be situated on OY, the equations are $x=0$, $y=b$, or the point is $(0, b)$. If the point be situated on both axes, it is, of course, their point of intersection, viz. the origin, which is characterised by the equations $x=0$, $y=0$.

Equation to the Straight Line.

7. If OM$=a$, and PM$=b$, the coordinates of every point in the unlimited line OP are in the ratio of $a : b$. Thus, by similar triangles,

$$\frac{\text{PM}}{\text{OM}}=\frac{\text{QL}}{\text{OL}}=\frac{\text{RN}}{\text{ON}}=\frac{b}{a}, \text{ or generally } \frac{y}{x}=\frac{b}{a}, \text{ or } y=\frac{b}{a}x;$$



where it should be observed that $\frac{b}{a}$ is an expression for the tangent of the angle which OP makes with the axis of x *in the positive direction*, that is, of the angular distance of OP from OX measured towards OY.

For the prolongation of OP downwards, if OM′ be taken $=-a$, then S′M′$=-b$, and the tangent of S′OX′$=\frac{-b}{-a}=\frac{b}{a}=$ tan POX.

It appears, then, that the equation $y=\frac{b}{a}x$ is satisfied by every point on the unlimited line S′OR.

If OM′$=-a$, and SM′$=b$, the coordinates of every point on the unlimited line OS are, as regards absolute magnitude, in the ratio of $a : b$; but if the expression for the ratio is to include reference to the line's position, we must now write $\frac{y}{x}=\frac{b}{-a}$ or $-\frac{b}{a}$, that is, $y=-\frac{b}{a}x$; which signifies that the tangent of the angle which SO makes with the axis of x, *in the positive direction* (viz. of the angle SOX, not SOX′), is $=-\frac{b}{a}$.

For the prolongation of SO downwards, we have OM$=a$, and P′M$=-b$ and the tangent of P′OX$=\frac{-b}{a}$ or $-\frac{b}{a}$, $=$ tan SOX.

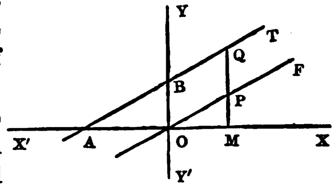
It appears, then, that the equation $y=-\frac{b}{a}x$ is satisfied by every point on the unlimited line P′OS.

8. *To find the general Equation to a Straight Line, referred to rectangular axes.*

The position of a straight line in relation to the axes is determined, when we know the inclination to OX of that part of the line which is above X′X, and also the distance of the line's intersection with YY′ from the origin O.

In the last article we found the *particular* equation characterising a straight line that intersects with YY′ at the origin. That equation may be written $y=mx$; where m denotes the tangent of the line's inclination to OX in the positive direction.

But, now, let AT be a straight line, of indefinite length, meeting the axis of y at B.

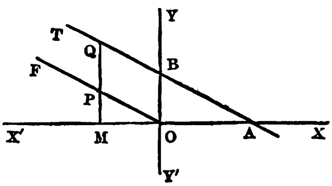


Draw OF, through the origin, parallel to AT. In AT take any point Q; and draw QM parallel to OY, meeting OX at M, and OF at P.

$$\frac{PM}{OM}=\tan FOX; \therefore PM=OM \tan FOX;$$

$$QM=PM+PQ=PM+OB;$$
$$\therefore QM=OM \tan FOX+OB.$$

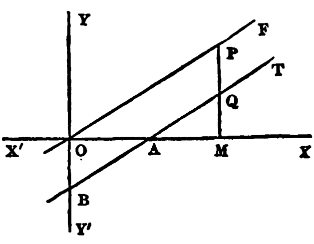
Hence, if OB, which is called the *intercept* on the axis of y, be given in length $=c$, and tan FOX, or tan TAX, $=m$, and if the coordinates of the point Q be x, y, then the equation to the straight line AT will be



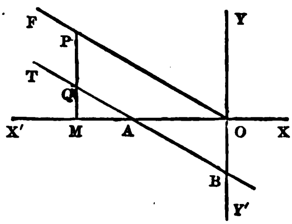
$$y=mx+c.$$

Suppose the line AT still to cut the axis of y above X'X, but to make the angle TAX obtuse; then the tangent of that angle will be negative, and m in the above equation stands for a negative quantity, c being still positive.

Suppose the line AT to cut the axis of y below X'X, and to make the angle TAX acute; then we have QM= PM−PQ=PM−OB, or, QM=OM tan FOX−OB; and in this instance the symbol m stands for a positive quantity, and c for a negative.



Lastly, suppose the line AT to cut the axis of y below X'X,



and to make the angle TAX obtuse; we shall then have m and c both standing for negative quantities.

Thus, then, if the absolute magnitude of m, in each of the particular cases supposed, were $=\frac{b}{a}$, and that of c were $=n$, the four varieties of lines would be represented as follows:

(i.) $y=\frac{b}{a}x+n.$ (ii.) $y=-\frac{b}{a}x+n.$

(iii.) $y=\frac{b}{a}x-n.$ (iv.) $y=-\frac{b}{a}x-n.$

So long as the same straight line, unrestricted as to length, is considered, the quantities m and c represent definite magnitudes, and are therefore called *constants*; while x, or y, is arbitrary, and may have an indefinite variety of values, to which y, or x, will have the same variety of *corresponding* values; the symbols x and y, therefore, are called *variables*.

9. Observe how the general equation

$$y=mx+c$$

provides for the following cases:

If the line pass through the origin of coordinates, then $c=0$, and the equation becomes

$$y=m\,x, \text{ as formerly found.}$$

If the line be parallel to the axis of x, it makes *no geometrical angle* with that axis; then $m=0$, and the equation becomes

$$y=c;$$

which signifies that the ordinate of *any* point on the line is $=c$.

If the line be parallel to the axis of y, then, when the

axes are rectangular, m is the tangent of a right angle, and is therefore infinite, and the required line will not cut off any intercept from the axis of y, so that c also is infinite, and if a be any abscissa, positive or negative, then $\frac{c}{a}$, which is also infinite, is equivalent to m; we shall thus have

$$y=\frac{c}{a}x+c, \text{ or } \frac{y}{\frac{c}{a}}=x+a, \text{ or } 0=x+a,$$

hence, as a may be positive or negative, the equation will be

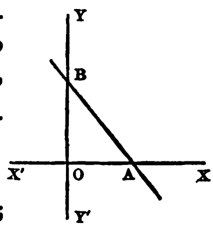
$$x=\pm a;$$

which signifies that the abscissa of any point on the line is $=a$.

The case, however, of a line parallel to the axis of y is very simply investigated by itself. For we need only say, Let a be the abscissa for *one* point on the required line, and x the abscissa of *any* point on the line; and we have $x=a$.

10. *The Equation to a Straight Line that makes intercepts on the axes may be expressed in terms of the intercepts.*

Let AB be a straight line cutting the axes at A and B; and let the intercepts OA, OB be respectively equal to a and b; then m, the tangent of BAX, $=-\frac{b}{a}$, and $c=\text{OB}=b$; and the equation



$$y=m\,x+c$$

becomes $y=-\frac{b}{a}x+b$, or $\frac{b}{a}x+y=b$;

then dividing by b gives

$$\frac{x}{a}+\frac{y}{b}=1, \text{ the required equation.}$$

11. It has now become evident that any straight line can be represented by an equation of the first degree.

We can prove also that any equation of the first degree will serve to represent a straight line.

For, an equation of the first degree being one into whose simple form there does not enter any power of an unknown quantity higher than the first power, or any product of unknown quantities, the general form of the indeterminate equation of the first degree is

$$A\,x+B\,y+C=0;$$

where A, B, C may be finite or zero.

When A is zero, we have $B\,y=-C$, or $y=-\frac{C}{B}$; which, as we have seen, represents a straight line parallel to the axis of x.

When B is zero, we have $A\,x=-C$, or $x=-\frac{C}{A}$; which, as we have seen, represents a straight line parallel to the axis of y.

If B is not zero, then, dividing by B, we have

$$\frac{A}{B}x+y+\frac{C}{B}=0, \quad \text{or, } y=-\frac{A}{B}x-\frac{C}{B}.$$

which, as we have seen, represents a line meeting the axis of y at a distance $=-\frac{C}{B}$ from the origin, and making with OX an angle whose tangent is $=-\frac{A}{B}$.

Therefore the equation $A\,x+B\,y+C=0$ will always serve to represent a straight line.

12. The following statements should now be distinctly understood:

The equation to a line is one that expresses an invariable relation which is satisfied by the coordinates of every point on the line.

The word *locus* (plural *loci*) is used to signify a lineal common-place or path, the coordinates of every point of which have the same ratio.

A line, whether straight or curved, is called the *locus* of a given equation, when the coordinates of every point on the line satisfy the equation.

13. It will now be a useful exercise for the student to draw the straight lines represented by a few given equations.

Example (1). It is required to draw the straight line whose position is represented by $2x-5y=7$.

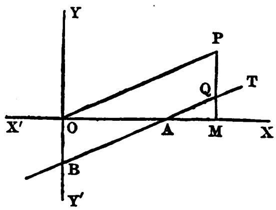
1*st Method.* Make the given equation assume the form

$$y=mx+c\,;$$

which is done by dividing by 5 and transposing; thus,

$$\tfrac{2}{5}x-y=\tfrac{7}{5}\,;\quad \therefore\ y=\tfrac{2}{5}x+(-\tfrac{7}{5}).$$

Here we have $m=\frac{2}{5}=$ the value of the tangent of the angle which that part of the required line which is above the axis of x will make with the axis of x produced in the positive direction.



First, then, draw the rectangular axes X′X, YY′; make PM perpendicular to OM, so that OM : PM as 5 : 2; and join OP. The angle POX is that of the required line's inclination to OX, because the tangent is $=\dfrac{\mathrm{PM}}{\mathrm{OM}}=\dfrac{2}{5}$. Now, any line parallel to OP will have the same inclination to OX. But the required line is to make on the axis of y a *negative* intercept whose absolute length is $\frac{7}{5}$. Accordingly, measuring $\mathrm{OB}=\frac{7}{5}$ downwards from O, and drawing through B a line parallel to OP, we have AB the required line, making with OX the positive angle TAX, whose tangent is $=\dfrac{\mathrm{QM}}{\mathrm{AM}}=\dfrac{\mathrm{PM}}{\mathrm{OM}}=\dfrac{2}{5}$.

2*nd Method.* Make the given equation assume the form

$$\frac{x}{a}+\frac{y}{b}=1\,;$$

which is done by dividing by 7, and then putting the coefficients of x and y in the form of divisors; thus,

$$2x-5y=7 \text{ will become } \tfrac{2}{7}x-\tfrac{5}{7}y=1;$$

$$\text{or, } \frac{x}{\frac{7}{2}}-\frac{y}{\frac{7}{5}}=1; \text{ or, } \frac{x}{\frac{7}{2}}+\frac{y}{-\frac{7}{5}}=1;$$

where $a=\frac{7}{2}$, $b=-\frac{7}{5}$.

If we take, therefore, from OX a positive abscissa, OA, whose absolute length is $\frac{7}{2}$, and from OY′ a negative ordinate, OB, whose absolute length is $\frac{7}{5}$, we shall have two points A and B on the required line, which are, of course, sufficient to determine the position of the line.

3rd Method. First, suppose $y=0$, and then find the corresponding value of x, to be used as an abscissa; secondly, suppose $x=0$, and find the corresponding value of y, to be used as an ordinate.

For, since the equation $2x-5y=7$ represents a line of indefinite length, one point of the line will meet the axis of x, and then, y being $=0$, we have

$$2x-0=7; \text{ or, } x=\tfrac{7}{2};$$

also, another point of the line will meet the axis of y, and then, x being $=0$, we have

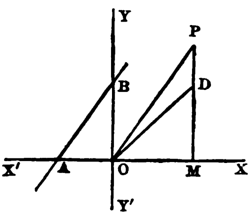
$$0-5y=7; \text{ or, } y=-\tfrac{7}{5}.$$

Note. The *2nd* and *3rd* methods are inapplicable in such a case as $x-2y=0$.

Ex. (2). Draw the straight line $y-x\sqrt{2}=5$.

Here, suppose $x=0$, then $y=5$; therefore take the positive ordinate OB$=5$.

Again, suppose $y=0$, then $x=-5\div\sqrt{2}=-\frac{5}{2}\sqrt{2}$; therefore take the negative abscissa OA$=\frac{1}{2}(5\sqrt{2})$, which is conveniently done thus: make OM$=$OB$=5$; draw MD perpendicular and equal to OM, and produce it to P, so that PM may be equal to the hypotenuse OD$=5\sqrt{2}$; then OA$=\frac{1}{2}$PM. Hence, to draw the required line, join AB.



EXERCISES [A].

Draw the lines represented by the following equations:

1. $3x-2y=4$.	2. $x+3=4y$.
3. $4x+5y=6$.	4. $\frac{1}{4}y-\frac{1}{3}x=5$.
5. $3x+4y+5=0$.	6. $x+y=0$.
7. $x\sqrt{3}-y=8$.	8. $x=-2$.
9. $\frac{4}{5}x+\frac{2}{3}=\frac{1}{2}y$.	10. $6y+x\sqrt{5}+\sqrt{10}=0$.

PROBLEMS ON THE STRAIGHT LINE.

14. *To find the Equation to a Straight Line that shall pass through a given point* (x', y')*.

Taking x and y as the coordinates of an indeterminate point on the line, let the line be represented by

$$y=mx+c. \qquad (1.)$$

The given coordinates x', y' must satisfy the general equation; therefore,

$$y'=mx'+c. \qquad (2.)$$

hence, eliminating c by subtraction, we have

$$y-y'=m\,(x-x'), \qquad (3.)$$

which is the required equation, and which may be written, if we choose, in the form of the general equation, thus:

$$y=mx+y'-mx';$$

where m is perfectly arbitrary.

It is evident that the required equation must reject either m or c; for it is in general impossible to draw a straight line through a given point so as to meet the axis of y at a proposed distance c from the origin, and at the same time make with the axis of x a proposed angle, $\tan^{-1}m$.

If we choose, m may be rejected and c retained. Thus, from equation (2) we have $m=\frac{y'-c}{x'}$, which will convert (1) into the required form

$$y=\frac{y'-c}{x'}x+c.$$

* Given coordinates of a point are very often expressed by the letters x and y accented.

15. *To find the equation to a straight line that shall pass through two given points,* (x', y') *and* (x'', y'').

Here we can assume the equations (1), (2), (3) from the preceding article:

$$y = mx + c, \qquad (1.)$$
$$y' = mx' + c, \qquad (2.)$$
$$y - y' = m(x - x') \qquad (3.)$$

(3) representing a line that fulfils one of the conditions now proposed: that is, a line passing through the point (x', y').

The line, however, is also to pass through (x'', y''), and, therefore,

$$y'' = m x'' + c. \qquad (4.)$$

Subtracting (2) from (4) we have

$$y'' - y' = m(x'' - x'),$$

$$\text{whence,} \quad m = \frac{y'' - y'}{x'' - x'};$$

m being thus no longer arbitrary.

Substituting in (3) the value of m just found:

$$y - y' = \frac{y'' - y'}{x'' - x'}(x - x') \qquad (5.)$$

is the required equation. It may be written, if we choose, in the form of the general equation, thus:

$$y = \frac{y'' - y'}{x'' - x'} x + \frac{y' x'' - x' y''}{x'' - x'}.$$

If we suppose $x'' = x'$, the line will be parallel to the axis of y; for the tangent will be $= \frac{y'' - y'}{0} = \infty$, $=$ the tangent of a right angle.

If we suppose $y'' = y'$, the line will be parallel to the axis of x; for the tangent becomes $= \frac{0}{x'' - x'} = 0$, $=$ the tangent of the trigonometrical angle which O X′ makes with O X.

If we suppose $x' = 0$, $y' = 0$, we make the point (x', y') the origin; for which case the equation is reduced to the simple form $y = \frac{y''}{x''} x$.

16. *To find the equation to a straight line which shall pass through a given point, and be parallel to a given straight line.*

Let the given line be represented by the equation

$$y=m'x+c', \qquad (1.)$$

where m' and c' are *known*;
and let the equation to the required line be

$$y=mx+c, \qquad (2.)$$

where m and c are *not known.*

Then, since the lines represented by these equations are parallel, they must have the same inclination to OX, that is, $m=m'$; thus the second equation becomes

$$y=m'x+c, \qquad (3.)$$

where c, the intercept on the axis of y, remains indeterminate, because an infinite number of lines may be parallel to a given line. The equation, in effect, asserts that it matters not what the intercept on the axis of y may be, provided the tangent of the inclination of the given line to the axis of x be also the tangent for any other line we may draw.

But, if the parallel be restricted to pass through a given point (x', y'), then

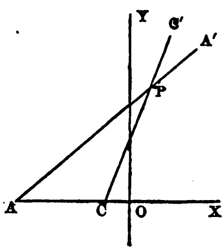
$$y'=m'x'+c; \qquad (4.)$$

Hence, subtracting (4) from (3), we have

$$y-y'=m'(x-x') \qquad (5.)$$

which is the required equation.

17. *To find an expression for the tangent of ϕ, the angle of intersection of two given straight lines.*



Let AA' and CC' be the two given lines, intersecting at P.

Let the equation to AA' be

$$y=m'x+c'$$

and the equation to CC'

$$y=m''x+c'';$$

m' is the tangent of the angle PAX;
m'' is the tangent of the angle PCX.

Now, $\tan \mathrm{APC}=\tan(\mathrm{PCX}-\mathrm{PAX})$

$$=\frac{\tan \mathrm{PCX}-\tan \mathrm{PAX}}{1+\tan \mathrm{PCX} \tan \mathrm{PAX}}=\frac{m''-m'}{1+m'm''};$$

$$\therefore \tan \text{ of supplement } \mathrm{A'PC}=\frac{m'-m''}{1+m'm''},$$

Hence, the required expression is $\tan\phi=\pm\frac{m''-m'}{1+m'm''}$, according as the angle is to the left or the right of CC′.

When the lines become parallel, the tangent $=0$, and therefore the numerator in the above expression $=0$, that is $m''=m'$.

In order that the angle of intersection may be a right angle, the tangent $=\infty$, and therefore the denominator in the above expression $=0$, or $m'm''=-1$, or $m''=-\frac{1}{m'}$.

Hence,
$$y=-\frac{1}{m'}x+c''$$
represents a line perpendicular to a given line whose position is expressed by $y=m'x+c'$.

Of course, there are an infinite number of such perpendiculars. Two lines intersect at right angles whenever the coefficient denoting the tangent in the equation to one line is the reciprocal, with a contrary sign, of that in the equation to the other line. In the next problem we fix one of the lines by the condition of its passing through a given point.

18. *To find the equation to a straight line that shall pass through a given point, and be perpendicular to a given straight line.*

Let the coordinates of the given point be x', y', and let the equation to the given line be

$$y=mx+c.$$

The equation to a line through (x', y') is of the form

$$y-y=m'(x-x');$$

and if this equation is perpendicular to the former, then $m' = -\frac{1}{m}$, and the required equation is

$$y-y' = -\frac{1}{m}(x-x').$$

19. *To find the equation to a straight line that passes through a given point (x', y'), and makes a given angle ϕ with a given straight line.*

Let the equation to the given line be

$$y = mx + c; \qquad (1.)$$

the equation to the required line passing through (x', y') will be of the form

$$y - y' = m'(x - x'). \qquad (2.)$$

By art. **17**, we have $\tan\phi = \pm\frac{m-m'}{1+mm'}$;

hence, $m' = \frac{m - \tan\phi}{1 + m\tan\phi}$ or $\frac{m+\tan\phi}{1-m\tan\phi}$;

therefore (2) becomes

$$y - y' = \frac{m-\tan\phi}{1+m\tan\phi}(x-x'). \qquad (3.)$$

or else,

$$y - y' = \frac{m+\tan\phi}{1-m\tan\phi}(x-x'). \qquad (4.)$$

Either of the equations (3) and (4) will be the one required; for generally through a given point two straight lines can be drawn, making a given angle with a given straight line.

Note (1). When $m = 0$, the given line is parallel to the axis of x; then the required equations are

$$y - y' = \pm\tan\phi\,(x - x').$$

Note (2). When $m = \infty$, the given line is parallel to the axis of y. For this case, dividing numerator and denominator of $\frac{m+\tan\phi}{1-m\tan\phi}$ by m, we have

$$\frac{1 + \frac{\tan\phi}{m}}{\frac{1}{m} - \tan\phi} = \frac{1 + \frac{\tan\phi}{\infty}}{\frac{1}{\infty} - \tan\phi} = \frac{1+0}{-\tan\phi} = -\frac{1}{\tan\phi}$$

Similarly, $\dfrac{m-\tan\phi}{1+m\tan\phi}$ becomes $=\dfrac{1}{\tan\phi}$.

Then the required equations are

$$y-y'=\pm\cot.\phi\,(x-x').$$

Note (3). Dividing numerator and denominator by tan ϕ, we have $\dfrac{\frac{m}{\tan\phi}\pm1}{\frac{1}{\tan\phi}\mp m}$, which, when ϕ is a right angle, is

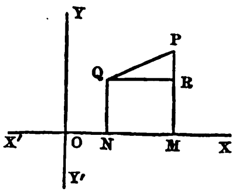
$$\frac{\frac{m}{\infty}\pm1}{\frac{1}{\infty}\mp m}=\frac{0\pm1}{0\mp m}=-\frac{1}{m},$$

as found in art 18.

Or thus, $\dfrac{\frac{m}{\tan\phi}\pm1}{\frac{1}{\tan\phi}\mp m}=\dfrac{m\cot\phi\pm1}{\cot\phi\mp m}=-\dfrac{1}{m}$;

the cotangent of a right angle being zero.

20. *To express the distance between two points in terms of their coordinates.*



Let P and Q be two points whose coordinates are given. Let the coordinates of Q be x', y', and let those of P be x'', y''. Join PQ; from P and Q draw PM, QN parallel to OY, and draw QR parallel to OX.

We have PRQ a right angle; therefore (by Euclid, I. 47)

$$PQ^2=QR^2+PR^2.$$

Now, $QR=OM-ON=x''-x'$; and $PR=PM-QN=y''-y'$; hence $\quad PQ^2=(x''-x')^2+(y''-y')^2$; which determines PQ.

And here it should be observed, that, as only the absolute length of PQ is required, we need take no notice of the double sign $\pm$ that would distinguish the algebraic roots of the value of PQ^2.

If the point Q were at the origin, we should have $x'=0$, $y'=0$; and then

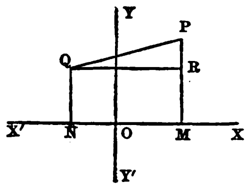
$$PQ^2=(x''-0)^2+(y''-0)^2=x''^2+y''^2.$$

And, if Q were on the axis of x, and P on that of y, we should have $x''=0$, $y'=0$; and then

$$PQ^2=(0-x')^2+(y''-0)^2=x'^2+y''^2.$$

Again, let P and Q be two points; let the given coordinates of Q be $-a$, b, and let those of P be h, k. Then, the expression for PQ^2, corresponding to that found above, will be $PQ^2=\{h-(-a)\}^2+(k-b)^2$,

$$=(h+a)^2+(k-b)^2$$

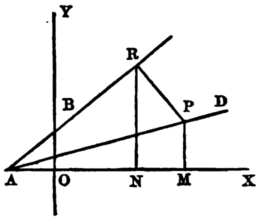


21. *To determine the distance of a given point from a given straight line.*

Let P be the given point, and AB the given line; it is required to determine the length of the perpendicular PR.

If the coordinates of P be h, k, and those of R be x_1, y_1, the analytical expression for the length of PR is

$$PR^2=(x_1-h)^2+(y_1-k)^2; \quad (1.)$$



Let the equation to the given line ABR be

$$y=mx+c;$$

then the equation to PR, perpendicular to ABR, is

$$y-k=-\frac{1}{m}(x-h);$$

and, as R is a point common to both these lines, we have

$$y_1-mx_1+c, \qquad (2.)$$

$$y_1-k=-\frac{1}{m}(x_1-h). \qquad (3.)$$

Multiplying (3) by m^2, and then adding (2), we have

$$(1+m^2)y_1-m^2k=-mh+c;$$

$$\text{whence, } y_1=\frac{m^2k+mh+c}{1+m^2}.$$

Multiplying (2) and (3) by m, and then subtracting, we get

$$mk=(1+m^2)x_1-h+mc;$$

$$\text{whence, } x_1=\frac{mk+h-mc}{1+m^2}.$$

Hence, by substitution in (1),

$$\text{PR}^2=\left(\frac{mk+h-mc}{1+m^2}-h\right)^2+\left(\frac{m^2k+mh+c}{1+m^2}-k\right)$$

$$=\left(\frac{mk+h-mc-h-m^2h}{1+m^2}\right)^2+\left(\frac{m^2k+mh+c-k-m^2k}{1+m^2}\right)^2$$

$$=\frac{m^2}{(1+m^2)^2}(k-mh-c)^2+\frac{1}{(1+m^2)^2}(k-mh-c)^2$$

$$=\frac{1}{1+m^2}(k-mh-c)^2;$$

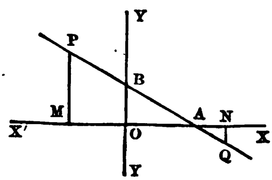
$$\therefore \text{PR}=\frac{k-mh-c}{\sqrt{(1+m^2)}}, \text{ the expression required.}$$

The positive and negative values of the denominator refer to the ambiguity of the direction of PR, as P may be on either side of the given line; but, as only the magnitude of PR is required, the denominator may be taken with the sign that will give a positive value for PR.

22. We shall now subjoin a few Examples and Exercises on articles **14–21.**

Examples.

Ex. (1). Draw, and represent by an equation, the straight line passing through the points P, Q, which are $(-4, 5)$ and $(7, -1\frac{1}{2})$, respectively. Find also the length of PQ.



Take OM, PM, $=4, 5$; ON, QN, $=7, 1\frac{1}{2}$; the line PQ, indefinitely produced, is the line required.

From art. **15**, we have

$$y-5=\frac{-1\frac{1}{2}-5}{7+4}(x+4),$$

$$\text{or, } y-5=-\tfrac{13}{22}(x+4),$$

which gives $13x+22y=58$, the required equation.

The equation determining the length PQ (*see* art. **20**)

$$\text{is } PQ^2=(-4-7)^2+(5+1\tfrac{1}{2})^2$$
$$=121+\tfrac{169}{4}=\tfrac{653}{4}\,;\ \therefore\ PQ=\tfrac{1}{2}\sqrt{653}.$$

Ex. (2). What will be the equation to a straight line passing through the point (21, 7) at right angles to the line $y-3x=24$?

(*See* art. **18.**) We have here $x'=21$, $y'=7$; and since the given line is represented by $y=3x+24$, we have $m=3$.

$$\text{Hence, } y-7=-\tfrac{1}{3}(x-21), \text{ or } 3y-21=-x+21,$$
$$\text{gives } x+3y=42, \text{ the required equation.}$$

Ex. (3). Shew that the straight lines

$$4x+3y=6, \qquad 5x+4y=7, \qquad 2x-y=8,$$

intersect at one point.

Here, by means of any two of the given equations, we can find values of x and y satisfying both. Take the 1st and 2nd:

$$\begin{aligned} 5x&=7-4y \\ 4x&=6-3y \\ \hline \end{aligned}$$

By subtraction, $x=1-y$

$$\therefore\ 5(1-y)=7-4y, \text{ or } y=-2, \text{ and } 1-y=x=3.$$

Therefore, the coordinates of the point of intersection of the 1st and 2nd lines are $x=3$, $y=-2$; and these will be found to be coordinates of a point on the 3rd line also; for $2x-y=6+2=8$.

Ex. (4). If the line $y=mx+7$ pass through the intersection of the lines $y=2x+6$ and $3y=x-7$, what is the magnitude of m?

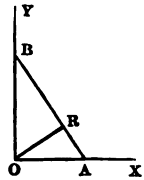
$$3y=6x+18=x-7, \text{ or } x=-5,$$
$$2x+6=y=-4\,;$$
$$\text{hence, } mx+7=-4,$$
$$\text{that is, } -5m+7=-4\,; \text{ or, } m=\tfrac{11}{5}.$$

Ex. (5). Find the distance of the origin of rectangular coordinates from the line $\dfrac{x}{a}+\dfrac{y}{b}=1$.

The given equation may be put in the form

$$y=-\frac{b}{a}x+b\,;$$

where we have $m=-\frac{b}{a}$; hence, applying the result found in art. **21**, we have $h=0$, $k=0$, $c=b$; and therefore



$$\text{OR}=\frac{k-mh-c}{\sqrt{(1+m^2)}}=-b\div\sqrt{\left(1+\frac{b^2}{a^2}\right)}$$
$$=-ab\div\sqrt{(a^2+b^2)}\,;$$

or the *magnitude* of the required distance is

$$=\frac{ab}{\sqrt{(a^2+b^2)}}.$$

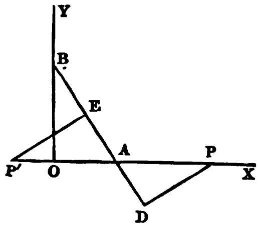
Note. Since the triangle OAB$=\frac{1}{2}$ the rectangle OA·OB, and also$=\frac{1}{2}$ the rectangle AB·OR, we have AB·OR$=$OA·OB;

but OA$=a$, OB$=b$, and AB$=\sqrt{(a^2+b^2)}$;

hence, OR$\sqrt{(a^2+b^2)}=ab$; and OR$=ab\div\sqrt{(a^2+b^2)}$; which verifies the formula.

Ex. (6). What points on the axis of x are at a perpendicular distance a from the line

$$\frac{x}{a}+\frac{y}{b}=1\,?$$



Take OA$=a$, OB$=b$; then AB is the given line.

Let PD, P′E be the perpendicular distances of AB from OX, and each $=a$.

$$\text{AB}=\sqrt{(a^2+b^2)}\,;\ \text{also, by similar triangles, } \frac{\text{AP}}{\text{PD}}=\frac{\text{AB}}{\text{OB}},$$

$$\text{whence, AP}=\frac{a}{b}\sqrt{(a^2+b^2)}.$$

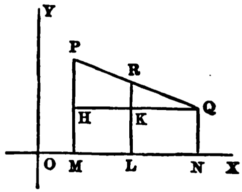
Therefore, the abscissa of the point P, $=$OA$+$AP, is

$$a+\frac{a}{b}\sqrt{(a^2+b^2)}\,;$$

and, as OP′ is on the negative side of O, therefore the abscissa of the point P′, $=OA-AP$, is

$$a-\frac{a}{b}\sqrt{(a^2+b^2)}.$$

Ex. (7). The coordinates of a point P are 4 and 10, and those of another point Q are 17 and 5; find the coordinates of a point R in the straight line PQ, making PR : QR :: 3 : 4.



Let x, y be the coordinates of R;

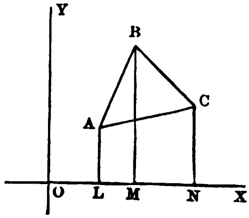
ML : NL :: PR : QR :: 3 : 4;

$$\frac{ML}{NL}=\frac{x-4}{17-x}=\frac{3}{4};\ \therefore x=9\tfrac{4}{7}.$$

Again, PQ : QR :: 7 : 4 :: HP : KR; $\therefore$ $KR=\frac{4}{7}HP=\frac{4}{7}(10-5)=2\frac{6}{7}$; hence $y=KR+QN=2\frac{6}{7}+5=7\frac{6}{7}$.

Therefore the coordinates of R are $9\frac{4}{7}$, $7\frac{6}{7}$.

Ex. (8). Find the area of a triangle, the coordinates of its angular points being 8 and 9, 13 and 21, 23 and 12, for A, B, C, respectively.



Here we have OL, AL= 8, 9;

OM, BM=13, 21;

ON, CN=23, 12.

Area of trapezium LABM

$$=\tfrac{1}{2}(AL+BM)LM=\tfrac{1}{2}(9+21)(13-8)=75$$

Area of trapezium MBCN

$$=\tfrac{1}{2}(BM+CN)MN=\tfrac{1}{2}(21+12)(23-13)=165$$

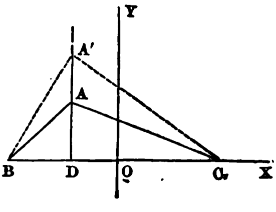
$$\therefore \text{Area of LABCN}=240$$

Area of trapezium $LACN=\frac{1}{2}(9+12)(23-8)=157\frac{1}{2}$

$$\therefore \text{Area of ABC}=82\tfrac{1}{2}$$

Ex. (9). Given BC, the base of a triangle, $=a$, and the difference of the squares of its sides$=d^2$, to determine the locus of the vertex.

Take the base as the axis of x, and its middle point O as the origin; and let the coordinates of the vertex A be x and y.



$BA^2 = BD^2 + DA^2 = (\frac{1}{2}a - x)^2 + y^2$,
$AC^2 = DC^2 + DA^2 = (\frac{1}{2}a + x)^2 + y^2$;
hence $AC^2 - BA^2 = 2ax = d^2$;

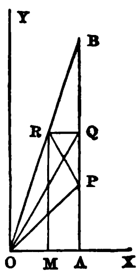
$$\therefore x = \frac{d^2}{2a}.$$

This is the equation to a straight line parallel to the axis of y (*see* art. **9**). Therefore the locus (*see* art. **12**) of the vertex A is a perpendicular to the base, at the distance $\frac{d^2}{2a}$ from the middle of the base.

Ex. (10). Find the equations to the diagonals of a quadrilateral whose sides are represented by the equations

$$y = x,\ y = 3x,\ x = 3,\ y = 5.$$

Take OA=3, AP=3, AQ=5, AB=9; join OP, OQ, OB; draw QR parallel to OA, and join PR. Then OPQR is the quadrilateral; for PQ is the line $x = 3$, also QR is the line $y = 5$, OR is represented by $y = 3x$, and OP by $y = x$.



$$\text{Now, } \frac{OM}{RM} = \frac{OA}{AB} = \frac{1}{3};\ \therefore OM = \tfrac{1}{3}RM = 1\tfrac{2}{3}.$$

Wherefore, the coordinates of R are $1\frac{2}{3}$, 5, and those of P are 3, 3; hence the equation to the diagonal PR is

$$y - 5 = \frac{3-5}{3-1\frac{2}{3}}(x - 1\tfrac{2}{3});$$

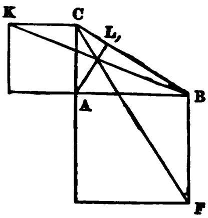
$$\text{or } y - 5 = -\tfrac{3}{2}(x - 1\tfrac{2}{3}), \text{ or } 2y - 10 = -3x + 5,$$

$$\text{which gives } 2y + 3x - 15 = 0.$$

Also, the coordinates of O are 0, 0, and those of Q are 3, 5; hence the equation to the diagonal QO is simply

$$3y = 5x.$$

Ex. (11). Shew that in the figure to Euclid, I. 47, the lines BK, CF, AL meet at one point.



Take A as the origin, AB, AC, produced indefinitely, as the axes of x and y.

Let AB$=a$, AC$=b$.

The equation to BC is $y=-\frac{b}{a}x+b$.

The equation to AL, passing through the origin, perpendicular to BC, is $y=\frac{a}{b}x$. (1)

The equation to BK, passing through the points $(-b, b)$ and $(a, 0)$, is $y-b=\frac{0-b}{a+b}(x+b)$,

$$\text{or, } y=-\frac{b}{a+b}x+\frac{ab}{a+b}. \qquad (2)$$

The equation to CF, passing through the points $(0, b)$ and $(a, -a)$, is $y-b=\frac{-a-b}{a}(x-0)$,

$$\text{or, } y=-\frac{a+b}{a}x+b. \qquad (3)$$

We are now to shew that the lines (1), (2), (3) intersect at one point.

The values of x and y satisfying (1) and (2) will be the coordinates of the point of intersection of these lines, and can be found thus:

$$y=\frac{a}{b}x=-\frac{b}{a+b}x+\frac{ab}{a+b}; \text{ or, } \left(\frac{a}{b}+\frac{b}{a+b}\right)x=\frac{ab}{a+b};$$

$$\text{or, } \frac{a^2+ab+b^2}{b}x=ab; \text{ whence } x=\frac{ab^2}{a^2+ab+b^2}.$$

$$\frac{a}{b}x=y=\frac{a^2b}{a^2+ab+b^2}.$$

Substituting this value of y in (3), we shall have, if the three lines intersect at one point,

$$\frac{a^2b}{a^2+ab+b^2}=-\frac{a+b}{a}x+b; \text{ or, } \frac{-b^2(a+b)}{a^2+ab+b^2}=-\frac{a+b}{a}x;$$

$$\text{or, } x=\frac{b^2}{a^2+ab+b^2}\div\frac{1}{a}=\frac{ab^2}{a^2+ab+b^2}, \text{ as before.}$$

EXERCISES [B].

1. Draw the straight line passing through the points $(3, -5)$ and $(-5, 2)$; and represent it by an equation.

2. Draw, and represent by an equation, the straight line passing through the points $(0, -12)$ and $(11, 7)$.

3. What are the equations to two straight lines, one of which passes through the origin, the other through the point $(13, 4)$, and both of them parallel to another line whose equation is $7y-3x=36$?

4. Draw, and represent by an equation, a straight line passing through the point $(5, -1)$, and perpendicular to the line $3y+5x=4$.

5. What are the equations to the straight lines passing through the point $(5, 30)$, and each inclined at an angle of $60°$ to the line $\frac{x}{32}+\frac{y}{14}=1$?

6. The coordinates of a point P are 5 and -4, and those of another point Q are -7 and 12. What is the length of the straight line PQ?

7. Find the distance between the points $(-12, 15)$ and $(16, 36)$.

8. Shew that the three lines

$$y=x+1, \quad 2y=3x+18, \quad 3y=2x-13,$$

meet at one point.

9. What number for m, in the equation $y=-mx+8$, would make that equation represent a line passing through the point of intersection of the lines

$$y=\tfrac{1}{2}x+3, \quad y=6x-12?$$

10. Find the equation to a straight line which passes through the origin, and also through the point of intersection of the lines

$$6x+7y+2=0, \quad 7x+9y-1=0.$$

11. The angular points of the triangle ABC are $(-21, -25)$, $(5, 26)$, and $(46, 0)$. Draw the figure, and find the area of the triangle.

12. ABC is a triangle: suppose the points A, B, C, to be,

respectively, (17, 20), (44, 33), (51, 8): Find the area of the triangle.

13. OPQ is a triangle: Take the point O as the origin of coordinates; then

(i.) Express the area of the triangle in terms of the coordinates of P and Q.

(ii.) Suppose the points P and Q to be respectively (5, 12) and (12, 9), and find the lengths of the three sides.

14. Find the points of intersection of the straight lines

$$x+3y=11, \quad 2x+y=12, \quad 5x+2y=16;$$

and shew that the area of the triangle formed by joining these points is $32\frac{1}{2}$.

15. The coordinates of A, B, C, the angular points of a triangle, are, respectively (3, 4), (8, 9), (10, 6). Find the equation to the line DE joining the middle points of AB and BC. Find also the inclination of DE to AC.

16. Find the equations to the diagonals of the rectangle formed by the four lines $x=a$, $x=a'$, $y=b$, $y=b'$.

17. Given $4y=3x$, and $x=12$, the respective equations to OP, PQ two sides of the parallelogram OPQR; and $12y=23x$, the equation to the diagonal OQ: Find the equations to the other sides and diagonal; and also the area of the parallelogram.

18. ABCD is a square, the rectangular coordinates of the points A and C being (2, 3) and (3, 5): find the equations to the four sides.

19. Find the equation to a straight line bisecting P, the vertical angle of the triangle OPQ; the equation to the side OP being $5y=12x$, and that to PQ being $\frac{x}{24}+\frac{y}{7}=1$.

20. In the figure of Euclid, I. 47, if KH and FG be produced to meet at P, and PA produced to meet BC at R: shew that PR is perpendicular to BC.

21. It is required to determine the sides and angles of the triangle ABC, the base AB and a perpendicular to it

through A being assumed as the axes of x and y, and the equations to BC and AC being, respectively,

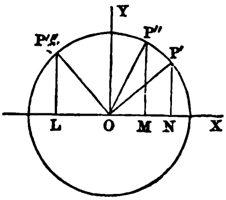
$$y=mx+c, \quad y=m'x.$$

Equation to the Circle.

23. We are now passing from the consideration of rectilineal to that of curvilineal loci; and begin with the circle, as the simplest of the regular curves.

An equation to the circle is one of which the circumference of a circle is the locus.

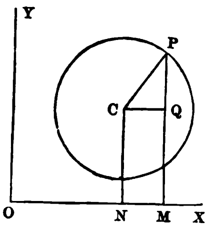
Now, the circumference of a circle will evidently be characterised by an expression denoting that every point in the curve is equidistant from a fixed point. If, in the annexed figure, OP′=OP″=OP‴, and the accented letters denote successive positions of the point P, then the locus of P is the circumference of a circle; and if O be the origin of coordinates of P, we have the sum of the squares of the coordinates constantly equal to the square of radius. Hence, if c be the radius of the circle,

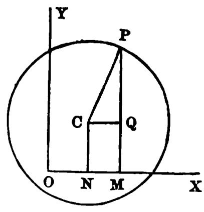


$$x^2+y^2=c^2$$

is an equation to the circle implying the centre to be the origin of coordinates.

24. Let us take for the origin of coordinates any point O in the plane of the circle.





Let C be the centre of the circle: its coordinates ON=a,

$NC=b$; and let P be any point on the circumference: its coordinates $OM=x$, $MP=y$. Draw CQ parallel to OX.

Then $CQ=OM-ON=x-a$; $PQ=PM-CN=y-b$.

$$\text{By Eucl. I. 47,} \quad CQ^2+PQ^2=CP^2;$$

$$\text{that is,} \quad (x-a)^2+(y-b)^2=c^2,$$

or, by development,

$$x^2+y^2-2ax-2by+a^2+b^2-c^2=0;$$

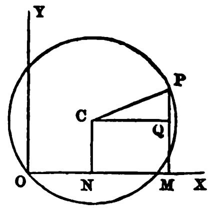
which is the general equation to the circle, referred to rectangular axes.

When the origin is a point on the circumference, then

$ON^2+NC^2=OC^2=c^2$, or $a^2+b^2-c^2=0$;

and the general equation gives

$$x^2+y^2-2ax-2by=0.$$



If OX coincide with CQ, and the origin be on the circumference, the centre C will be the point $(a, 0)$, and a will be $=c$, and $\therefore a^2-c^2=0$; then the general equation gives $\qquad x^2+y^2-2ax=0.$

If OY coincide with NC, and the origin be on the circumference, the centre C will be the point $(0, b)$, and b will be $=c$, and $\therefore b^2-c^2=0$; then the general equation gives

$$x^2+y^2-2by=0.$$

But the centre is most frequently taken as the origin; in which case $a=0$, $b=0$; then the general equation gives

$$x^2+y^2=c^2, \text{ as in art. } \mathbf{23}.$$

25. It is evident, then, that the general equation to the circle, for rectangular axes, is of the form

$$x^2+y^2+Ax+By+C=0;$$

which is an indeterminate equation of the 2nd degree, having unity as the coefficient of each of the squares of x and y, and not containing the product of these variables.

And, conversely, it may be shewn that the locus of an equation of the above form is always the circumference of

a circle; although it will include the peculiarities of a circumference reduced to a mere point, and of a curve merely imaginary.

For, making x^2+Ax and y^2+By complete squares, we may write the above equation thus:

$$x^2+Ax+\tfrac{1}{4}A^2+y^2+By+\tfrac{1}{4}B^2=\tfrac{1}{4}A^2+\tfrac{1}{4}B^2-C;$$

$$\text{that is, } (x+\tfrac{1}{2}A)^2+(y+\tfrac{1}{2}B)^2=\tfrac{1}{4}(A^2+B^2-4C);$$

which, if compared with the form

$$(x-a)^2+(y-b)^2=c^2,$$

will be observed to represent a circle the coordinates of whose centre are $-\frac{1}{2}A$ and $-\frac{1}{2}B$, and of which the radius is $\frac{1}{2}\sqrt{(A^2+B^2-4C)}$.

The circumference will be reduced to a mere point, viz. the centre, if $A^2+B^2=4C$, for then the sum of the squares on the left side of the equation will be zero. It is more accurate to say here that the circumference will be reduced to one whose radius is indefinitely small.

The locus of the equation will be impossible if $4C$ be greater than A^2+B^2; for then $\sqrt{(A^2+B^2-4C)}$ will be the square root of a negative quantity, which is merely imaginary.

26. The preceding article indicates an easy method of constructing any circle that is represented by a given equation. Suppose, for instance, it is required to construct the circle whose equation is

$$x^2+y^2+12x-10y-60=0.$$

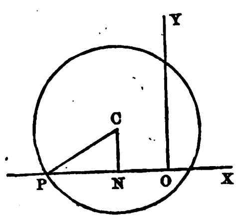
Completing squares with x^2+12x and y^2-10y, we have

$$(x+6)^2+(y-5)^2=60+6^2+5^2$$
$$=121;$$

and comparing this with the form

$$(x-a)^2+(y-b)^2=c^2,$$

we see that $a=-6$, $b=5$, $c=11$. Accordingly, the circle will be represented, in magnitude and position, as in the annexed figure, where ON is taken $=6$, NC$=5$, CP$=11$.



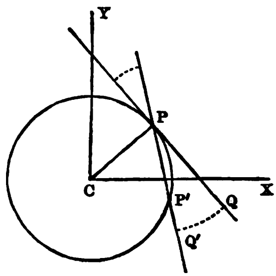
Tangent and Normal to a Circle.

27. A secant to a circle is a straight line meeting the circumference at two points. Suppose one of the points, at which a secant meets the circumference, to remain fixed, while the secant turns round that point and shifts the second point up to the first, the secant in its limiting position is called the tangent to the circle at the first point. (*See* figure to art. **28.**)

The tangent may thus be regarded as a secant passing through *two coincident points* of the circumference; and this consideration suggests an easy method of investigating the equation that will represent the tangent to a circle.

28. *To find the equation to the tangent at a given point on the circumference of a circle.*

Take the centre as origin, that is, let $x^2+y^2=c^2$ be the equation to the circle; and let x', y' be the coordinates of the point P, on the circumference, at which the tangent is drawn, x'', y'' the coordinates of another point P′ on the circumference, near the given point.



The equation to the straight line PP′ is

$$y-y'=\frac{y''-y'}{x''-x'}(x-x'). \qquad (1.)$$

$$\text{Now, } x'^2+y'^2=c^2\text{; also, } x''^2+y''^2=c^2\text{;}$$

$$\therefore\ y'^2-y''^2=x''^2-x'^2,$$

$$\text{that is, } (y'+y'')(y'-y'')=(x''+x')(x''-x')\text{;}$$

$$\therefore\ \frac{y'-y''}{x''-x'}=\frac{x'+x''}{y'+y''}\text{; or, } \frac{y''-y'}{x''-x'}=-\frac{x'+x''}{y'+y''}\text{;}$$

hence, by substitution in the equation to PP′,

$$y-y'=-\frac{x'+x''}{y'+y''}(x-x'). \qquad (2.)$$

Now, when the point P′, by the revolution of the straight line P′P round P, has shifted its position on the curve until

the chord PP′ becomes indefinitely small, that is, when the point (x'', y'') coincides with the point (x', y'), we have $x''-x'=0$, and $y''-y'=0$; or $x''=x'$, and $y''=y'$; in which case equation (2) becomes

$$y-y'=-\frac{2x'}{2y'}(x-x');$$

or, the equation to the tangent at the point P, that is, at the point (x', y'), is

$$y-y'=-\frac{x'}{y'}(x-x'). \qquad (3.)$$

It will now be seen that for the case in which $x''=x'$ and $y''=y'$, equation (1) would have given an indeterminate expression, because that equation does not include the condition of (x', y') and (x'', y'') being points on the circumference of a circle.

The equation to the radius CP is, of course,

$$y=\frac{y'}{x'}x;$$

compare it with (3): the tangent number in one is the reciprocal, with contrary sign, of that in the other, thus shewing that a tangent to a circle is always perpendicular to the radius passing through the point of contact.

Equation (3) may be changed to a simpler form; for by reduction we have

$$yy'-y'^2=-xx'+x'^2,$$
$$\text{or, } xx'+yy'=x'^2+y'^2,$$
$$\text{that is, } xx'+yy'=c^2.$$

This is the form in which the equation to the tangent is most frequently referred to.

When the equation to the circle is in the form

$$(x-a)^2+(y-b)^2=c^2,$$

the equation to the tangent at any point (x', y') may be similarly found to be

$$(x-a)(x'-a)+(y-b)(y'-b)=c^2.$$

29. *To express the tangent to a circle in terms of the tangent of the angle which the line makes with the axis of x.*

We have seen that the tangent of the angle which QP,

the tangent to the circle, makes with the axis of x is $=-\frac{x'}{y'}$.

The equation $yy'+xx'=c^2$

$$\text{gives } y=-\frac{x'}{y'}x+\frac{c^2}{y'}; \qquad (1.)$$

and the equation $x'^2+y'^2=c^2$

$$\text{gives } \frac{x'^2}{y'^2}+1=\frac{c^2}{y'^2}, \text{ or, } \frac{c^2}{y'}=c\sqrt{\frac{x'^2}{y'^2}+1};$$

whence, by substitution in (1), we have

$$y=-\frac{x'}{y'}x+c\sqrt{\frac{x'^2}{y'^2}+1};$$

or, writing m for $-\frac{x'}{y'}$,

$$y=mx+c\sqrt{(1+m^2)}; \qquad (2.)$$

which is the equation to the tangent in terms of the tangent of the angle which the line makes with the axis of x.

Hence, if the general equation to a straight line be

$$y=mx+n,$$

then, to determine the condition that any straight line should touch the circle, we have

$$mx+n=mx+c\sqrt{(1+m^2)},$$
$$\text{or, } n^2=c^2(1+m^2).$$

30. *To find the equation to the normal at any point on the circumference of a circle.*

The *normal* to a curve is the line intersecting the tangent at right angles at the point of contact.

Let (x', y') be the point of contact of a tangent to a circle. The equation to the tangent is

$$y=-\frac{x'}{y'}x+\frac{c^2}{y'}.$$

The equation to the normal as a straight line passing through (x', y') is of the form.

$$y-y'=m'(x-x');$$

and the condition of its being perpendicular to the tangent

requires that m' shall denote the reciprocal, with contrary sign, of the m found in the equation to the tangent; or requires that $m' = -\frac{1}{m} = \frac{y'}{x'}$; thus the equation to the normal is

$$y - y' = \frac{y'}{x'}(x - x'),$$

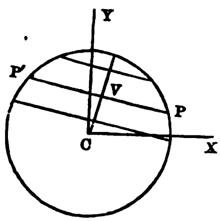
or by reduction, $x'y = xy'$, or $y = \frac{y'}{x'}x$;

which is the equation to a straight line passing through the origin, that is, through the centre.

The equation to the normal is anticipated in art. **28**; where it is shewn that a tangent to a circle is always perpendicular to the radius passing through the point of contact.

31. *To find the locus of the middle points of a system of parallel chords.*

Let PP′ be one of the chords; x', y' the coordinates of P; and x'', y'' those of P′.



The equation to PP′ is

$$y - y' = \frac{y'' - y'}{x'' - x'}(x - x').$$

Now, because P and P′ are points on the circumference,

$$c^2 = x'^2 + y'^2 = x''^2 + y''^2;$$

$$\therefore x''^2 - x'^2 = y'^2 - y''^2,$$

$$\text{or, } (x'' + x')(x'' - x') = (y' + y'')(y' - y'');$$

$$\therefore \frac{y'' + y'}{x'' + x'} = -\frac{x'' - x'}{y'' - y'}.$$

But the coordinates of V, the middle point of PP′, are $\frac{1}{2}(x'' + x')$ and $\frac{1}{2}(y'' + y')$; so that the tangent of the angle which a line through that point and the origin makes with the axis of x is

$$\frac{y'' + y'}{x'' + x'} = -\frac{x'' - x'}{y'' - y'}$$

=the reciprocal, with contrary sign, of the tangent number in the equation to PP′; hence, the locus of V is a line through the centre perpendicular to the chords (*See* Euclid, III. 3).

32. *To find the equation to a straight line that shall touch a given circle, and pass through a given point external to the circle.*

Let h, k be the coordinates of the given point; and let x', y' (which will presently become known) be the coordinates of the point of contact.

The equation to the tangent at (x', y') is

$$xx'+yy'=c^2;$$

and, as the tangent passes through the given point (h, k), the equation to the tangent must be satisfied by the coordinates of that point; and, therefore,

$hx'+ky'=c^2$; which is the required equation; but the point of contact being on the circumference, we have

$$x'^2+y'^2=c^2;$$

so that x' and y' are now determined, and will each have two values; therefore *two* tangents can be drawn to the circle from the given point; and

$$hx+ky=c^2$$

will be the general equation to the line joining the two points of contact, that is, to the *chord of contact*, as that line is called.

33. We shall now subjoin a few Examples and Exercises on articles **23–32**.

Examples.

Ex. (1). Find the points of intersection of the circle $x^2+y^2=4$ with the straight line $3x+5y=6$.

The 2nd equation gives $9x^2=36-60y+25y^2$.

The 1st equation gives $9x^2=36-9y^2$.

By subtraction, $34y^2-60y=0$;

$$\text{hence, } y=0, \text{ or } \tfrac{30}{17};$$

$$x=2-\tfrac{5}{3}y=2, \text{ or } -\tfrac{16}{17}.$$

Therefore the straight line meets the circle at the points $(2, 0)$ and $(-\frac{16}{17}, \frac{30}{17})$; the former of these points being an extremity of the diameter in the axis of x.

Ex. (2). Find the points of intersection of the circle $x^2+y^2-26x-4y+73=0$ with the line $12x+5y=216$.

The 2nd eqn. gives $25y^2=144x^2-5184x+46656$.

„ 1st „ $25y^2=-25x^2+650x-1825+100y$.

By subtraction, $169x^2-5834x+48481=100y$,

that is, from 2nd eqn. $=4320-240x$;

hence, $169x^2-5594x=-44161$;

which gives $13x-\frac{2797}{13}=\pm\frac{600}{13}$.

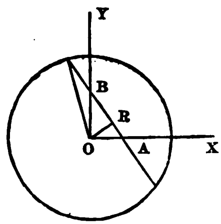
Accordingly, we have $x=13$, or $20\frac{17}{169}$;

$$y=12, \text{ or } -5\tfrac{7}{169};$$

and, therefore, the straight line meets the circle at the points $(13, 12)$ and $(20\frac{17}{169}, -5\frac{7}{169})$; the former of these being an extremity of the diameter parallel to the axis of y, since the abscissa, 13, is that of the centre, the equation to the circle being

$$(x-13)^2+(y-2)^2=10^2.$$

Ex. (3). Find the length of the chord made by the circle $x^2+y^2=c^2$ with the line $\frac{x}{a}+\frac{y}{b}=1$.



Let the intercepts on the axes be OA $=a$, OB$=b$, $\therefore$ AB$^2=a^2+b^2$; and OR, the perpendicular on AB, is determined, by similar triangles, thus:

$$AB^2 : OB^2 :: OA^2 : OR^2=\frac{a^2b^2}{a^2+b^2};$$

Now, OR bisects the chord, and therefore half the chord $=\sqrt{c^2-\dfrac{a^2b^2}{a^2+b^2}}$;

$\therefore$ the length of the chord is $=2\sqrt{c^2-\dfrac{a^2b^2}{a^2+b^2}}$.

Ex. (4). Find the equation to the circle described on a diameter of which the extremities are (x', y') and (x'', y'').

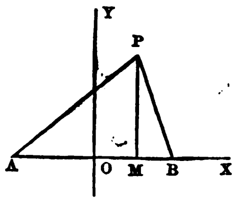
Here the coordinates of the centre are $\frac{1}{2}(x'+x'')$ and $\frac{1}{2}(y'+y'')$, therefore,

$$\left\{x-\tfrac{1}{2}(x'+x'')\right\}^2+\left\{y-\tfrac{1}{2}(y'+y'')\right\}^2=\text{rad.}^2$$
$$=\tfrac{1}{4}(x''-x')^2+\tfrac{1}{4}(y''-y')^2;$$

or $x^2-(x'+x'')x+\frac{1}{4}(x''+x')^2+y^2-(y'+y'')y+\frac{1}{4}(y''+y')^2$
$$-\tfrac{1}{4}(x''-x')^2-\tfrac{1}{4}(y''-y')^2=0$$

or, $x^2+y^2-(x'+x'')x-(y'+y'')y+x'x''+y'y''=0$; the equation required.

Ex. (5). Given AB the base of a triangle, and the ratio of its sides AP, BP$=m$: shew that the locus of the vertex P is a circle, except when $m=1$.



Take the base AB as the axis of x, and O the middle point of AB as origin. Let x, y be the coordinates of the vertex P; and put AB$=2a$.

If AP$=m$ BP, then AP$^2=m^2$BP^2;

$$\text{or, AM}^2+\text{PM}^2=m^2(\text{BM}^2+\text{PM}^2);$$

that is, $(a+x)^2+y^2=m^2\left\{(a-x)^2+y^2\right\}$;

$$\text{or } m^2(x^2+y^2-2ax+a^2)=x^2+y^2+2ax+a^2;$$

that is,

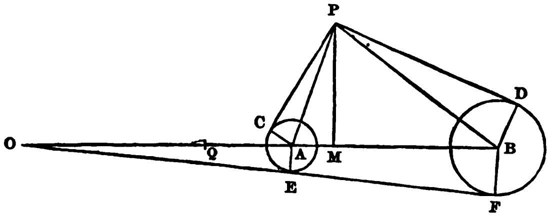
$$(m^2-1)x^2+(m^2-1)y^2-2(m^2+1)ax+(m^2-1)a^2=0;$$

hence, $$x^2+y^2-2\cdot\frac{m^2+1}{m^2-1}ax+a^2=0;$$

which is the equation to a circle, unless $m=1$; for when $m=1$ the equation becomes $x^2+y^2-\infty+a^2=0$; then we should have $x=0$, and y indefinite or arbitrary, and BP$^2=$ $y^2+a^2=$AP^2; in which case the locus would be the axis of y.

Ex. (6). In one plane, the locus of the point from which two given unequal circles would appear equally large, that is, subtend equal angles, is a circle.

Let AC the radius of the smaller circle $=r$, BD that of the larger $=r'$, and AB the distance of the centres $=a$. Let P be a point from which the circles subtend equal angles, viz. doubles of APC, BPD; PC and PD being tangents.



Draw EF a common tangent to the circles, and let it meet AB at O. Take O as origin, x, y as the coordinates of P.

By similar triangles, $\frac{\text{OA}}{\text{OB}-\text{OA}}=\frac{\text{AE}}{\text{BF}-\text{AE}}$; hence, if we put d for OA, we have $\frac{d}{a}=\frac{r}{r'-r}$, or $d=\frac{ar}{r'-r}$.

$$\text{AM}=x-d,\ \text{BM}=a+d-x,\ \text{PM}=y.$$

By similar triangles, $\frac{\text{PA}^2}{\text{PB}^2}=\frac{\text{AC}^2}{\text{BD}^2}=\frac{r^2}{r'^2}$;

$$\text{that is, } \frac{(x-d)^2+y^2}{(d+a-x)^2+y^2}=\frac{r^2}{r'^2};$$

$$\text{whence } (x^2+y^2)(r'^2-r^2)-2x\{r'^2d-r^2(d+a)\}$$

$$=r^2(d+a)^2-r'^2d^2, =0, \text{ since } d+a=\frac{ar'}{r'-r};$$

$$\text{moreover, } r'^2d-r^2(d+a)=\frac{arr'^2-ar'r^2}{r'-r}=arr';$$

$$\therefore x^2+y^2-2x\,.\,\frac{arr'}{r'^2-r^2}=0,$$

which is the equation to a circle, whose centre is on the axis of x at a point Q equidistant from the points O and P; the radius OQ=PQ $\frac{arr'}{r'^2-r^2}$.

EXERCISES [C].

1. Determine the radius, and the coordinates of the centre, for each of the following circles:

(i.) $x^2+y^2-6x-8y-24=0$.

(ii.) $x^2+y^2+2x+4y-4=0$.

(iii.) $x^2+y^2-6x-18y+10=0$.

(iv.) $x^2+y^2+x-2y-3\frac{3}{4}=0$.

2. Find the points of intersection of the circle $x^2+y^2=34$:

(i.) with the line $x-y=2$.

(ii.) with the line $2x+y=-1$.

3. Find the points of intersection of the circle $x^2+y^2=1$:

(i.) with the line $3x-4y=5$.

(ii.) with the line $5x+4y=-5$.

4. Find the points of intersection of the line

$$4y-3x+32=0:$$

(i.) with the circle $x^2+y^2-24x-10y=0$.

(ii.) with the circle $2x^2+2y^2-8x+y-45=0$.

5. A straight line parallel to the line $y=mx+n$ touches the circle $(x-a)^2+(y-b)^2=c^2$. Find the equation to the tangent.

6. Find the length of the chord which the line $12y-5x=56$ makes with the circle $x^2+y^2-40x=0$.

7. What must be the relation between the quantities a, b, c, in order that the line $\frac{x}{a}+\frac{y}{b}=1$ may touch the circle $x^2+y^2=c^2$?

8. Two circles, whose radii are c, c', cut one another; the distance between their centres being a. Find the coordinates of the points of intersection.

9. From the equation to the circle, which is the locus of P in the 5th of the worked Examples, viz.,

$$x^2+y^2-2\,.\,\frac{m^2+1}{m^2-1}ax+a^2,$$

determine the position of the centre, the length of the radius, and the points at which the circle cuts the axis of x.

10. Shew that the square of half the base of a triangle and the square of the line joining the vertex and the middle of the base are together equal to half the sum of the squares of the two sides of the triangle. Shew also that, when the base and the sum of the squares of the sides are constant, the locus of the vertex is a circle.

11. Find the equation to a circle whose centre is at the origin of coordinates, and which touches the line $y=3x+5$.

12. Find the equation to the circle whose diameter is the common chord of the circles $x^2+y^2=c^2$ and $(x-a)^2+y^2=c^2$, where a denotes the distance between the two centres on the axis of x.

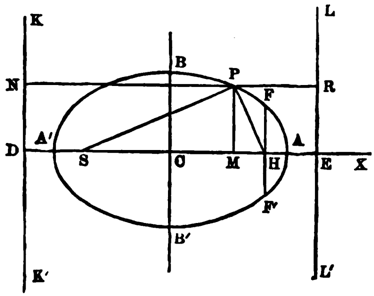
13. What must be value of c, in order that the circles $(x-a)^2+(y-b)^2=c^2$ and $(x-b)^2+(y-a)^2=c^2$ may touch each other?

14. Suppose the equation to a circle whose radius is c to be $x^2+y^2-2cx=0$, and the straight line joining its points of intersection with the line $y=mx$ to have a circle described on it as diameter. Required the equation to the latter circle.

THE ELLIPSE.

34. The Ellipse is the locus of a point whose distance from a given fixed point is always less, in a constant ratio, than its distance from a given fixed straight line.

Let S be the given point, and KK′ the given line. Draw



SD perpendicular to KK′. Let P be a point on the locus; join SP, and draw PM parallel to KK′ and PN parallel to DS.

Take D as the origin, DS as the axis of x, KK′ as the axis of y.

Divide DS at A′ so that $\frac{A'S}{A'D}=\frac{SP}{PN}$ = the constant ratio, a value less than unity, which we may call e. Let DS be called p; and let x, y be the coordinates of P.

We have $SP=e \cdot PN$; $\therefore SP^2=e^2PN^2$; or $PM^2+SM^2=e^2DM^2$; that is, $y^2+(x-p)^2=e^2x^2$,

which is the equation to the ellipse with the assumed origin and axes.

35. For the point where the ellipse meets the axis of x, we have $y=0$, and the equation becomes

$$(x-p)^2=e^2x^2;$$

but this gives $x-p=\pm ex$, or $(1\mp e)x=p$, or $x=\frac{p}{1\mp e}$;

signifying that the ellipse meets the axis at two points, A′ and A; so that $DA' = \frac{p}{1+e}$, and $DA = \frac{p}{1-e}$.

C the middle point of A′A is called the centre, and the extremities A′, A are called the vertices of the ellipse. Also, the fixed point S is called the focus, and the fixed line KK′ the directrix of the ellipse.

36. Let the line $A'A = 2a$; then $A'C = a$.

Since A is a point on the locus,

we have $e = \frac{SA}{DA} = \frac{2a - A'S}{2a + DA'} = \frac{2a - e \cdot DA'}{2a + DA'}$;

$$\therefore a = ae + e \cdot DA'.$$

Whence are derived the following values:

(i.) $DA' = \frac{a}{e}(1-e)$;

(ii.) $A'S = e \cdot DA' = a(1-e)$;

(iii.) $DC = a + DA' = \frac{a}{e}$;

(iv.) $SC = a - A'S = ae$;

(v.) $DS = p = DC - SC = \frac{a}{e} - ae = \frac{a}{e}(1-e^2)$.

37. Suppose the origin to be at A′.

In this case A′M takes the place of DM; and if A′M be called x', then $DM = DA' + x'$; that is, $x = x' + \frac{a}{e} - a$; whence, by substitution in the equation $y^2 + (x-p)^2 = e^2x^2$, we have

$$y^2 + (x' + \frac{a}{e} - a - \frac{a}{e} + ae)^2 = \{ex' + a(1-e)\}^2;$$

$$\text{or } y^2 + \{x' - a(1-e)\}^2 = \{ex' + a(1-e)\}^2;$$

$$\text{or } y^2 + x'^2 - 2a(1-e)x' = e^2x'^2 + 2ae(1-e)x';$$

$$\text{or } y^2 + x'^2 - 2a(1+e)(1-e)x' = e^2x'^2;$$

$$\text{or } y^2 + x'^2(1-e^2) = 2ax'(1-e^2);$$

$$\text{or } y^2 = (1-e^2)(2ax' - x'^2).$$

Now, as A is to be accounted the origin, we may remove the accent, and the equation will be

$$y^2=(1-e^2)(2ax-x^2).$$

38. Suppose the origin to be at C.

Here again we might put x' for CM, and substitute $x'+\frac{a}{e}$ for x in the original equation.

Or we may at once let x and y be the coordinates; then we have

$$SM=CM+SC=x+ae;$$

$$DM=CM+DC=x+\frac{a}{e},$$

$$\text{or } e\cdot DM=ex+a;$$

$\therefore$, equivalent to $PM^2+SM^2=e^2DM^2$, we have

$$y^2+(x+ae)^2=(ex+a)^2;$$

$$\text{or } y^2=(1-e^2)(a^2-x^2).$$

Now, let b denote the ordinate BC; then for the coordinates of the point B we have $x=0$, $y=b$, and the above equation gives us $b^2=a^2(1-e^2)$, or $1-e^2=\frac{b^2}{a^2}$. The equation, therefore, may be written

$$y^2=\frac{b^2}{a^2}(a^2-x^2),$$

which is reducible to the more symmetrical form

$$a^2y^2+b^2x^2=a^2b^2, \text{ or } \frac{x^2}{a^2}+\frac{y^2}{b^2}=1;$$

which is the equation to the ellipse referred to the centre as origin, and is the one most frequently employed.

39. The above equation is convertible into the forms

$$y=\pm\frac{b}{a}\sqrt{a^2-x^2}, \qquad (1)$$

$$\text{and } x=\pm\frac{a}{b}\sqrt{b^2-y^2}, \qquad (2)$$

which enable us readily to determine the form of the ellipse.

If $y=0$, then (2) gives $x=\pm a$; therefore the ellipse cuts the axis of x at the points A, A′.

If x be supposed a greater magnitude than $+a$ or $-a$, then in (1) we shall have a^2-x^2 negative, and the values of y are imaginary. Therefore no point on the ellipse can have a greater abscissa than a or $-a$.

If x be supposed a less magnitude than $+a$ or $-a$, then (1) shews that for every value of x there are two values of y, numerically equal, but contrary in sign. This implies that for every point above the axis of x there is a corresponding point at the same distance below the axis of x; so that the axis of x divides the curve symmetrically.

Also from (2) it is evident that the curve is symmetrical with respect to the axis of y, and that no point on the ellipse has an ordinate exceeding b.

Accordingly, the ellipse is a continuous curve returning into itself, and divided by the axis of x into two equal parts; and it is easily shewn that *the centre of an ellipse is a point that bisects every chord of the ellipse drawn through that point.*

It may be here stated that the term *axes*, in reference to the ellipse, is generally used to denote the lines A′A and BB′ intercepted by the curve; and that the former is called the major axis or transverse axis, the latter the minor or conjugate axis.

It may be also mentioned that the ratio denoted by the symbol e is called the *eccentricity* of the ellipse.

40. The ellipse being symmetrically situated with respect to the axes, A′A, BB′, there must be two foci, each having its corresponding directrix; and if AH=A′S, and AE=DA′, the curve can be described by means of H and LL′, as focus and directrix.

41. The readiest mechanical method of describing an ellipse is furnished by the following property:

The sum of the focal distances of any point on the ellipse is equal to the axis major.

$$\text{For } SP = e \cdot PN = e(DC + CM) = e(\frac{a}{e} + x) = a + ex;$$

$$\text{and } HP = e \cdot PR = e(CE - CM) = e(\frac{a}{e} - x) = a - ex;$$

$$\therefore SP + PH = 2a = A'A.$$

Hence, for the point B we have $SB = A'C$.

Accordingly, an ellipse may be constructed as follows:

Let a string SPH be fastened to two points S and H, then, if a pencil P be moved so as to keep the string always stretched, it will trace an ellipse of which S and H are the foci, since $SP + PH$ will be constantly $= A'A$.

This property of the constant sum of the focal distances is sometimes stated as a definition of the ellipse (*See* Ex. 4, p. 54).

42. Either focus divides the axis major into segments, the rectangle of which is equal to the square of the semi-axis minor, so that $A'S \cdot SA = BC^2$.

For $A'C^2 = SB^2 = SC^2 + BC^2$; therefore $BC^2 = A'C^2 - SC^2 = (A'C - SC)(A'C + SC) = A'S \cdot SA$.

Or thus: $A'S = a(1-e)$, and $SA = a(1+e)$; and we have already found (38) the product of these, viz. $a^2(1-e^2), = b^2$.

43. The equation $y^2 = \frac{b^2}{a^2}(a^2 - x^2)$ may be put in the form

$$y^2 = \frac{b^2}{a^2}(a+x)(a-x);$$

hence, if PM be any ordinate, and C the origin,

$$PM^2 = \frac{BC^2}{A'C^2} A'M \cdot MA,$$

$$\text{or } PM^2 : A'M \cdot MA :: BC^2 : A'C^2.$$

44. The double ordinate FHF′ passing through the focus is called the *latus rectum*. To find its length, we have $x = CH = SC = ae$, which substituted for x in the equation

$y^2=\frac{b^2}{a^2}(a^2-x^2)$ gives

$$y^2=\frac{b^2}{a^2}(a^2-a^2e^2)=b^2(1-e^2)=b^2\cdot\frac{b^2}{a^2}=\frac{b^4}{a^2};$$

$$\text{hence } y=\text{FH}=\frac{b^2}{a}; \text{ and FHF}'=\frac{2b^2}{a}.$$

TANGENT AND NORMAL TO AN ELLIPSE.

45. The tangent to an ellipse may be regarded as a secant passing through two coincident points on the curve. (*See* art. **27**.)

To find the equation to the tangent at a given point on an ellipse.

Let x', y' be the coordinates of P the given point, x'', y'' those of another point P′ on the curve, near the given point.

The equation to the straight line PP′ is

$$y-y'=\frac{y''-y'}{x''-x'}(x-x'). \qquad (1)$$

The equation to the ellipse, referred to the centre as origin, is

$$a^2y^2+b^2x^2=a^2b^2.$$

Accordingly, since (x', y') and (x'', y'') are points on the curve

$$a^2y'^2+b^2x'^2=a^2b^2,$$

$$a^2y''^2+b^2x''^2=a^2b^2;$$

$$\therefore a^2(y''^2-y'^2)=-b^2(x''^2-x'^2),$$

$$\text{or, } a^2(y''+y')(y''-y')=-b^2(x''+x')(x''-x');$$

$$\therefore \frac{y''-y'}{x''-x'}=-\frac{b^2}{a^2}\cdot\frac{x''+x}{y''+y'};$$

hence, the 1st equation, by substitution, will become

$$y-y'=-\frac{b^2}{a^2}\cdot\frac{x''+x'}{y''+y'}(x-x'). \qquad (2)$$

Now, when the point (x'', y'') coincides with the point (x', y'), we have $x''=x'$ and $y''=y'$; in which case (2) becomes

$$y-y'=-\frac{b^2}{a^2}\cdot\frac{2x'}{2y'}(x-x'),$$

that is, $$y-y'=-\frac{b^2x'}{a^2y'}(x-x'),$$

$$\text{or } a^2yy'+b^2xx'=a^2y'^2+b^2x'^2,$$

$$\text{that is, } a^2yy'+b^2xx'=a^2b^2. \qquad (3)$$

46. *To express the tangent to an ellipse in terms of the tangent of the angle which the line makes with the axis of x.*

We have seen that the tangent of the angle which the line makes with the major axis is $-\frac{b^2x'}{a^2y'}$.

From the equation $a^2yy'+b^2xx'=a^2b^2$

we have $$y=-\frac{b^2x'}{a^2y'}x+\frac{b^2}{y'}; \qquad (1)$$

and from the equation $a^2y'^2+b^2x'^2=a^2b^2$

we have $$1+\frac{b^2x'^2}{a^2y'^2}=\frac{b^2}{y'^2};$$

$$\therefore \frac{b^2}{y'}=\pm b\sqrt{\frac{b^2x'^2}{a^2y'^2}+1}. \qquad (2)$$

Let $-\frac{b^2x'}{a^2y'}=m$; whence $\frac{x'}{y'}=-\frac{a^2m}{b^2}$;

$$\therefore \frac{x'^2}{y'^2}=\frac{a^4m^2}{b^4}; \text{ and } \frac{b^2x'^2}{a^2y'^2}=\frac{a^2m^2}{b^2};$$

hence, by substitution in (2),

$$\frac{b^2}{y'}=\pm b\sqrt{\frac{a^2m^2}{b^2}+1}; \text{ or, } \frac{b^2}{y'}=\pm\sqrt{a^2m^2+b^2}.$$

Whence, by substitution in (1),

$$y=mx\pm\sqrt{a^2m^2+b^2}; \qquad (3)$$

which is the equation to the tangent in terms of the tangent of the angle which the line makes with the major axis of the ellipse; the double sign implying that *two* tangents may be drawn making angles of equal magnitude with the axis of x. Hence, if the general equation to a straight line be

$$y=mx+n,$$

then, to determine the condition that any straight line should touch the ellipse, we have

$$mx+n=mx+\sqrt{a^2m^2+b^2},$$
$$\text{or } n^2=a^2m^2+b^2.$$

47. *To find the equation to the normal at any point of an ellipse, and its intercepts on the axes.*

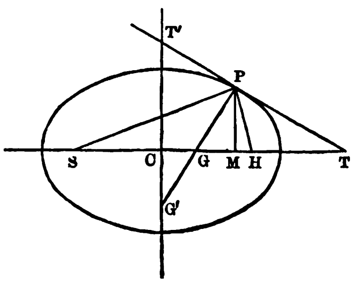
The *normal*, as formerly defined, is the line intersecting the tangent at right angles at the point of contact.

Let (x', y') be the point of contact of a tangent to the ellipse. The equation to the tangent is

$$y=-\frac{b^2x'}{a^2y'}x+\frac{b^2}{y'}.$$

The equation to the normal as a straight line through (x', y') is of the form $y-y'=m'(x-x')$; and the condition of its being perpendicular to the tangent requires that m' shall denote the reciprocal, with contrary sign, of the m found in the equation to the tangent, or requires that $m'=-\frac{1}{m}=\frac{a^2y'}{b^2x'}$; hence the equation to the normal at (x', y') is

$$y-y'=\frac{a^2y'}{b^2x'}(x-x').$$



Now, at G, where the normal PG cuts the axis major, $y=0$, and the above equation gives

$$-y'=\frac{a^2y'}{b^2x'}(x-x'); \text{ or, } -1=\frac{a^2}{b^2x'}(x-x');$$

$$\text{or, } x=x'\left(1-\frac{b^2}{a^2}\right)=e^2x';$$

that is, $CG=e^2CM$.

Again, at G′, where the normal cuts the axis minor, $x=0$, and the equation to the normal gives

$$y-y'=\frac{a^2y'}{b^2x'}(0-x'); \text{ or, } y=y'-\frac{a^2y'}{b^2};$$

$$\text{or, } y=y'\left(-1\frac{a^2}{b^2}\right)=\frac{a^2e^2}{b^2}y';$$

that is, $CG'=\frac{a^2e^2}{b^2}PM$.

Note. The length of PG in terms of the focal distances of P and the eccentricity may be determined as follows:—

$$SP=a+ex'; \ HP=a-ex'; \ SM=ae+x'.$$

$$\therefore PM^2=SP^2-SM^2=(a^2-x'^2)(1-e^2);$$

$$GM=CM-CG=x'(1-e^2);$$

$$GM^2=x'^2(1-e^2)^2.$$

$$\text{Hence } PG^2=PM^2+GM^2=(1-e^2)\{a^2-[1-(1-e^2)]x'^2\}$$
$$=(1-e^2)(a^2-e^2x'^2)=(1-e^2)(a+ex')(a-ex');$$
$$\text{or } PG^2=(1-e^2)SP\cdot HP.$$

48. *The focal distances of any point on the ellipse make equal angles with the tangent at that point.*

C being the origin, SP and HP the focal distances of P, and $SC=CH=ae$; the coordinates of S are $-ae$, 0, and those of H are ae, 0: let those of P be x', y'.

The equation to SP is

$$y-0=\frac{y'-0}{x'+ae}(x+ae),$$

showing the value of tan PSM to be $\frac{y'}{x'+ae}$.

The equation to HP is

$$y-0=\frac{y'-0}{x'-ae}(x-ae),$$

showing the value of tan PHT to be $\dfrac{y'}{x'-ae}$;

$$\therefore \text{ that of tan PHM is } -\frac{y'}{x'-ae}.$$

The equation to the tangent at P is

$$y-y'=-\frac{b^2x'}{a^2y'}(x-x'),$$

showing the value of tan PTM to be $\dfrac{b^2x'}{a^2y'}$.

Tan SPT′=tan(PSM+PTM)

$$=\frac{\dfrac{y'}{x'+ae}+\dfrac{b^2x'}{a^2y'}}{1-\dfrac{b^2x'}{a^2(x'+ae)}}=\frac{a^2y'^2+b^2x'^2+ab^2ex'}{(a^2-b^2)x'y'+a^3ey'}$$

$$=\frac{a^2b^2+ab^2ex'}{a^2e^2x'y'+a^3ey'}=\frac{b^2(a+ex')}{aey'(a+ex')}=\frac{b^2}{aey'}.$$

Tan HPT=tan (PHM−PTM)

$$=\frac{-\dfrac{y'}{x'-ae}-\dfrac{b^2x'}{a^2y'}}{1-\dfrac{b^2x'}{a^2(x'-ae)}}=\frac{a^2y'^2+b^2x'^2-ab^2ex'}{(b^2-a^2)x'y'+a^3ey'}$$

$$=\frac{a^2b^2-ab^2ex'}{-a^2e^2x'y'+a^3ey'}=\frac{b^2(a-ex')}{aey'(a-ex')}=\frac{b^2}{aey'}.$$

Therefore the angle SPT′=the angle HPT. And because PG is perpendicular to TT′ we have also the complements SPG, HPG equal to each other; so that the angle between the focal distances of any point on the ellipse is bisected by the normal at that point.

The above is a rigorous analytical proof of the proposed theorem. The following is another demonstration, assuming the 2nd part of Euclid VI. 3.

In (**47**) we found CG$=e^2x'$;

$\therefore$ SG$=$SC$+$CG$=ae+e^2x'=e(a+ex')=e.$SP (art. 41)

$$HG = CH - CG = ae - e^2x' = e(a - ex') = e \cdot HP \quad \text{(art. 41)}$$

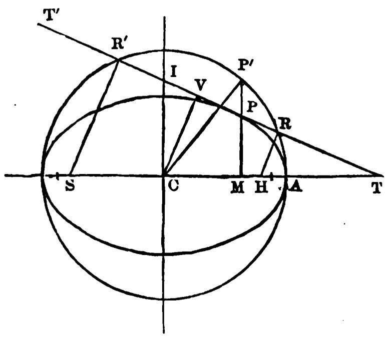
$$\therefore \frac{SG}{HG} = \frac{SP}{HP};$$

hence (Eucl. VI. 3) SPG=HPG; and ∴ SPT′=HPT.

49. *If a circle be described on the axis major as diameter, and any ordinate,* MP, *of the ellipse be produced to meet the circle at* P′, *then* $\frac{PM}{P'M} = \frac{b}{a}$.

For $CM^2 + P'M^2 = CA^2 = a^2$, or, $a^2 - CM^2 = P'M^2$; and the equation to the ellipse gives $\frac{CM^2}{a^2} + \frac{PM^2}{b^2} = 1$, or, $a^2 - CM^2 = \frac{a^2}{b^2} PM^2$. Therefore $\frac{a^2}{b^2} PM^2 = P'M^2$,

$$\text{or } \frac{PM}{P'M} = \frac{b}{a}.$$



The angle P′CM is called the eccentric angle of the ellipse.

50. *To find the locus of the intersection of a tangent to the ellipse with the perpendicular drawn to it from either focus.*

The equation to RR′ (*see* figure in **49**) is

$$y - mx = \sqrt{a^2m^2 + b^2}. \qquad (1)$$

The equation to SR′ drawn from the focus whose coordinates are $-ae$, 0, is of the form

$$y=m'(x+ae);$$

and that this line may be perpendicular to the former we put $m'=-\frac{1}{m}$, or

$$y=-\frac{1}{m}(x+ae);$$

$$\text{or, } my+x=-ae. \qquad (2)$$

Similarly, the equation to HR drawn perpendicular from the focus whose coordinates are ae, 0, will be

$$y=-\frac{1}{m}(x-ae),$$

$$\text{or } my+x=ae. \qquad (3)$$

Squaring (1) and (2), or (1) and (3):

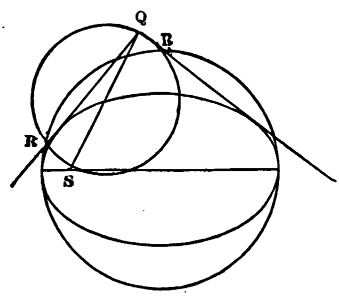
$$y^2-2myx+m^2x^2=a^2m^2+b^2;$$
$$m^2y^2+2myx+x^2=a^2e^2.$$

By addition

$$y^2(1+m^2)+x^2(1+m^2)=a^2m^2+b^2+a^2e^2$$
$$=a^2m^2+a^2=a^2(1+m^2);$$
$$\therefore y^2+x^2=a^2,$$

the equation to a circle whose radius is a. Therefore the required locus is a circle of which the major axis of the ellipse is a diameter.

Note. The preceding result suggests the following geometrical method of drawing a tangent to an ellipse from any external point Q.



Let a circle be described on the major axis as diameter, and another on the line SQ as diameter; then straight lines, drawn from Q through R, R′, the points of intersection of the two circles, will be the two tangents to the ellipse.

For the angles QRS, QR'S in a semicircle are right angles, or R, R' are points on the locus of the intersection of a perpendicular from the focus with the tangent.

51. *To determine the perpendicular* (p) *from the centre of an ellipse as origin, to the tangent at any point on the curve, in terms of the angle* θ *which* p *makes with the axis of* x.

(*See* the fig. in art. **49.**) $CV=p$; $VCT=\theta$.

The equation to the straight line TV is

$$\frac{x}{a}+\frac{y}{b}=1;$$

multiplying by p, we have

$$\frac{p}{a}x+\frac{p}{b}y=p;\ \text{that is,}\ \frac{CV}{CT}x+\frac{CV}{CI}y=p;$$

but $\dfrac{CV}{CT}=\cos\theta$, and $\dfrac{CV}{CI}=\sin\theta$;

$$\therefore\ x\cos\theta+y\sin\theta=p;$$

which, it should be remembered, is a convenient expression for a straight line in terms of the perpendicular on it from the origin and the inclination of this perpendicular to the axis of x. It gives

$$y=-\frac{\cos\theta}{\sin\theta}x+\frac{p}{\sin\theta};$$

and the condition that this line should touch the ellipse is

$$\frac{p^2}{\sin^2\theta}=a^2\frac{\cos^2\theta}{\sin^2\theta}+b^2;\ \text{or}\ p^2=a^2\cos^2\theta+b^2\sin^2\theta;$$

$$\text{or}\ p^2=a^2\cos^2\theta+a^2(1-e^2)\sin^2\theta;$$

$$=a^2(\cos^2\theta+\sin^2\theta)-a^2e^2\sin^2\theta;$$

$$\text{that is,}\ p^2=a^2(1-e^2\sin^2\theta);$$

which determines p as required.

52. In this article we shall give a few Examples and Exercises on the Ellipse.

EXAMPLES.

Ex. (1). Find the eccentricity of the ellipse

$$5x^2+8y^2=2n^2.$$

Here, in order to make the equation assume the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

we divide it by $2n^2$, which gives

$$\frac{x^2}{\frac{2}{5}n^2}+\frac{y^2}{\frac{1}{4}n^2}=1\,;$$

$$\therefore\ a^2=\tfrac{2}{5}n^2,\ b^2=\tfrac{1}{4}n^2,\ \text{and}\ \frac{b^2}{a^2}=\tfrac{5}{8}.$$

Now $e^2=1-\dfrac{b^2}{a^2}=1-\tfrac{5}{8}=\tfrac{3}{8}=\tfrac{1}{16}$ of 6;

$$\therefore\ e=\tfrac{1}{4}\sqrt{6}.$$

Ex. (2). Find the eccentricity of the ellipse in which the normal at F, the extremity of the latus rectum, passes through B′, the extremity of the axis minor.

From art. **47** we have $\text{CB}'=\dfrac{a^2e^2}{b^2}\text{FH}$;

$$\text{that is, } b=\frac{a^2e^2}{b^2}\cdot\frac{b^2}{a}=ae^2\,;$$

$$\therefore\ e^4=\frac{b^2}{a^2}=1-e^2\,;\ \text{or}\ e^4+e^2=1\,;$$

$$\text{whence } e=\tfrac{1}{2}\sqrt{(2\sqrt{5}-2)}.$$

Ex. (3). The focal distances of a point P, on an ellipse, are SP=18, HP=8, and the semi-axis minor is 5. Find the coordinates of P referred to the centre as origin.

(*See* the fig. in **34.**) $\quad a=\tfrac{1}{2}(18+8)=13\,;\ b=5\,;$

hence $e=\sqrt{\left((1-\dfrac{b^2}{a^2}\right)}=\tfrac{12}{13}$; also $\text{SC}=ae=12$, $\text{SH}=24$,

$$\text{SM}^2-\text{MH}^2=\text{SP}^2-\text{HP}^2\,;$$

that is, $(\text{SM}+\text{MH})(\text{SM}-\text{MH})=(\text{SP}+\text{HP})(\text{SP}-\text{HP})$,

$$\text{or } 24(\text{SM}-\text{MH})=26\times 10\,;\ \therefore\ \text{SM}-\text{MH}=10\tfrac{5}{6}\,;$$

$$\text{hence SM}=17\tfrac{5}{12}\,;$$

$$\therefore\ \text{CM}=\text{SM}-\text{SC}=5\tfrac{5}{12}.$$

$$\text{PM}^2=(a^2-x'^2)(1-e^2)=(169-\tfrac{4225}{144})\times\tfrac{25}{169},$$

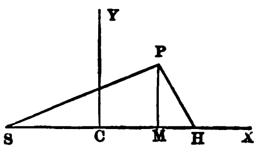
$$=25-\tfrac{25}{144}\text{ of }25=\tfrac{119}{144}\text{ of }25=\tfrac{25}{144}\text{ of }119\,;$$

$$\therefore\ \text{PM}=\tfrac{5}{12}\sqrt{119}\,;$$

therefore the required coordinates are $x'=5\frac{5}{12}$, $y'=\frac{5}{12}\sqrt{119}$.

Ex. (4). Investigate the equation to the curve that is the locus of a point, the sum of whose distances from two given points is constant.

Let S, H be the two fixed points, and take C the middle point of SH as origin. Let P be a point on the locus, and let



SP + HP = $2a$, and SH = $2c$.

$$\text{SP}^2=(c+x)^2+y^2,$$

$$\text{HP}^2=(c-x)^2+y^2,$$

$$\therefore\ \text{SP}^2-\text{HP}^2=4cx\,;$$

$$\frac{\text{SP}^2-\text{HP}^2}{\text{SP}+\text{HP}}=\text{SP}-\text{HP}=\frac{4cx}{2a}=\frac{2cx}{a}\,;$$

$$\text{hence, SP}=a+\frac{cx}{a}.$$

$$\therefore\ \left(a+\frac{cx}{a}\right)^2=(c+x)^2+y^2\,;$$

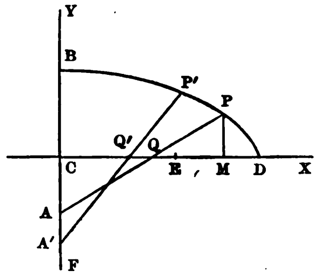
$$\text{which gives } a^2+\frac{c^2x^2}{a^2}=c^2+x^2+y^2\,;$$

$$\text{whence } y^2=\frac{a^2-c^2}{a^2}(a^2-x^2)\,;$$

which is the equation to an ellipse having $a^2-c^2=b^2$, that is, having S and H for the foci.

Ex. (5). Shew that the locus of one end of a given straight rod, whose other end and a given point in it move in straight lines at right angles to each other, is an ellipse.

Let AP be the given rod, Q the given point in it, P the end whose locus is required.



Let FY and CX be the two straight lines intersecting at right angles at C; and on them take segments FC, CB, respectively =CE, ED, =AQ, QP.

Let AP=CD=a, PQ=BC=b, CM=x, PM=y, and the angle PQM=θ.

By similar triangles, $\frac{\text{CM}}{\text{AP}}=\frac{\text{QM}}{\text{QP}}$, that is, $\frac{x}{a}=\cos\theta$;

$$\frac{\text{PM}}{\text{QP}} \quad \text{or} \quad \frac{y}{b}=\sin\theta;$$

$$\therefore \frac{x^2}{a^2}+\frac{y^2}{b^2}=\cos^2\theta+\sin^2\theta=1;$$

which is the equation to an ellipse.

Ex. (6). Determine the locus of the point of intersection of two tangents to an ellipse at right angles to each other.

The equation to the tangent is

$$y-mx=\pm\sqrt{a^2m^2+b^2};$$

if the tangent pass through a given point, (h, k), then

$$k-mh=\pm\sqrt{a^2m^2+b^2},$$

$$\text{or } m^2+\frac{2hk}{a^2-h^2}m+\frac{b^2-k^2}{a^2-h^2}=0; \qquad (1)$$

whence two values of m are determined; let them be called t and t'. Then, since

$$y-mx=k-mh, \text{ or } y-k=m(x-h),$$

the equations to the two tangents will be

$$y-k=t(x-h), \qquad (2)$$

$$y-k=t'(x-h); \qquad (3)$$

and if (2) and (3) are to be at right angles to each other, then

$$t'=-\frac{1}{t}, \text{ or } tt'+1=0;$$

but if $m=t$ or t', then $m-t=0$, and $m-t'=0$,

$$\therefore (m-t)(m-t')=0,$$

$$\text{or } m^2-(t+t')m+tt'=0,$$

which compared with (1) shows $tt'=\dfrac{b^2-k^2}{a^2-h^2}$;

$$\therefore \frac{b^2-k^2}{a^2-h^2}+1=0, \text{ or } b^2-k^2=h^2-a^2,$$

$$\text{or } h^2+k^2=a^2+b^2;$$

hence the locus of the point (h, k) is a circle.

Ex. (7). Find, in terms of the focal distances of a poi P on the ellipse, the perpendiculars HR, SR′ from the f on the tangent at P; and shew that $\text{HR}\cdot\text{SR}'=b^2$.

(*See* the figures in arts. **47**, **49**.)

The equation to the tangent at P is

$$a^2yy'-b^2xx'=-a^2b^2.$$

For the point T, where it meets the axis of x, we have

$$y=0, \therefore x=\text{CT}=\frac{a^2}{x'}; \text{ and } \text{GT}=\text{CT}-\text{CG}=\frac{a^2}{x'}-e^2x'$$

$$=\frac{a^2-e^2x'^2}{x'}; \text{ also } \text{HT}=\text{GT}-\text{GH}=\frac{a^2-e^2x'^2}{x'}-e(a-ex$$

$$=\frac{a(a-ex')}{x'}.$$

Now, by similar triangles, $\dfrac{\text{HR}^2}{\text{PG}^2}=\dfrac{\text{HT}^2}{\text{GT}^2}$,

$$\text{or, } \frac{\text{HR}^2}{(1-e^2)(a^2-e^2x'^2)}=\frac{a^2}{(a+ex')^2}; \text{ or, } \frac{\text{HR}^2}{\frac{b^2}{a^2}(a-ex')}=\frac{a^2}{a+ex'}$$

that is, $\dfrac{a^2}{b^2}\cdot\dfrac{\text{HR}^2}{\text{HP}}=\dfrac{a^2}{\text{SP}}$;

$$\text{hence, } \text{HR}^2=\frac{\text{HP}}{\text{SP}}\cdot b^2.$$

Similarly, it may be shewn that the other perpendicul is determined by $\quad \text{SR}'^2=\dfrac{\text{SP}}{\text{HP}}\cdot b^2.$

$$\therefore \text{HR}\cdot\text{SR}'=b^2.$$

EXERCISES [D].

1. Find the eccentricity of each of the following ellipses:

(i.) $3x^2+4y^2=c^2$. (ii.) $3x^2+5y^2=n^2$.

(iii.) $25x^2+169y^2=780^2$.

2. What is the eccentricity of the ellipse, when A′B is parallel to CF, in fig. of art. **34**?

3. The focal distances of a point P on an ellipse, whose eccentricity is $\frac{4}{5}$, are SP=4, HP=6. Find the coordinates of P referred to the centre as origin.

4. Find the equation to the normal at F, an extremity of the latus rectum.

5. The coordinates of a point P, on an ellipse whose eccentricity is $\frac{4}{9}\sqrt{2}$, are $x'=\frac{27}{8}\sqrt{2}$ and $y'=\frac{7}{8}\sqrt{238}$. Find SP and HP.

6. Shew that the equation to the tangent to the ellipse $2x^2+3y^2=18$, inclined to the axis of x at an angle of 30°, is $3y=x\sqrt{3}\pm 9$.

7. Shew that, in an ellipse, if $a^2=2b^2$, the angle SBH is a right angle.

8. Shew that the lengths of the perpendiculars from the foci of an ellipse, on a tangent inclined to the axis major at an angle ϕ, are $=a\{e\sin\phi\pm(1-e^2\cos^2\phi)^{\frac{1}{2}}\}$.

9. Assuming πa^2 as the area of the circle described on the major axis of the ellipse, shew that the radius of a circle is a mean proportional between the semi-axes of an ellipse equal in area to the circle.

10. Find the equation to the tangent at F, the extremity of the latus rectum of the ellipse $\frac{x^2}{9a^2}+\frac{y^2}{4a^2}=1$, and the lengths of the intercepts of the tangent on the axes.

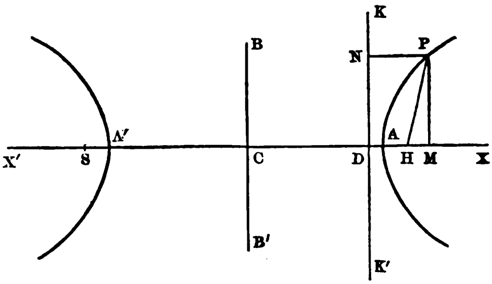
11. Two given circles touch each other internally: shew that the locus of the centre of a circle in the space between their circumferences, and touching both, is an ellipse having their centres for its foci.

12. Shew that if y be the ordinate of any point P on an ellipse, the tangent of the angle A′PA is $=-\frac{2b^2}{ae^2y}$.

THE HYPERBOLA.

53. The hyperbola is the locus of a point, whose distance from a given fixed point is always greater, in a constant ratio, than its distance from a given fixed straight line.

Let H be the given point, and KK′ the given straight line. Draw HD perpendicular to KK′. Let P be a point on the locus; join HP, and draw PM parallel to KK′, and PN parallel to DH.



Take D, in the line KK′, as the origin, DH as the axis of x, KK′ as the axis of y.

Divide DH at A, so that $\frac{AH}{AD}=\frac{HP}{PN}=$ the constant ratio, a value greater than unity, which we may call e. Let DH be called p; and let x, y be the coordinates of P.

We have $\quad HP=e\cdot PN$; $\therefore\ HP^2=e^2PN^2$,

$$\text{or, } PM^2+HM^2=e^2DM^2,$$

$$\text{that is, } y^2+(x-p)^2=e^2x^2;$$

which is the equation to the hyperbola referred to the origin and axes assumed.

54. For the point where the hyperbola meets the axis of x, we have $y=0$, and the equation becomes

$$(x-p)^2=e^2x^2;$$

this gives $x-p=\pm ex$, or $(1\mp e)x=p$, or $x=\frac{p}{1\mp e}$, signi-

fying that the hyperbola meets the axis at two points, A and A′; but as e exceeds *unity*, $1-e$ is a negative quantity; so that we have DA′, measured to the left of D, $=\dfrac{p}{e-1}$, and $\text{DA}=\dfrac{p}{1+e}$.

C, the middle point of A′A, is called the centre, and the extremities A, A′ are called the vertices, of the hyperbola. Also, the fixed point H is called the focus, and the fixed line KK′ the directrix of the hyperbola.

55. Let the line $\text{A}'\text{A}=2a$; then $\text{CA}=a$.

Since A′ is a point on the locus, we have

$$e=\frac{\text{HA}'}{\text{DA}'}=\frac{2a+\text{AH}}{2a-\text{DA}}=\frac{2a+e\,.\,\text{DA}}{2a-\text{DA}};$$

whence are derived the following values:

(i.) $\text{DA}=\dfrac{a}{e}(e-1)$;

(ii.) $\text{AH}=e\cdot\text{DA}=a(e-1)$;

(iii.) $\text{CD}=a-\text{DA}=\dfrac{a}{e}$;

(iv.) $\text{CH}=a+\text{AH}=ae$;

(v.) $\text{DH}=p=\text{CH}-\text{CD}=ae-\dfrac{a}{e}=\dfrac{a}{e}(e^2-1)$.

56. Suppose the origin to be at A.

In this case, AM takes the place of DM; and if AM be called x', then $\text{DM}=\text{DA}+x'$, that is, $x=x'+a-\dfrac{a}{e}$; whence, by substitution in the equation

$$y^2+(x-p)^2=e^2x^2,$$

we have $y^2+\left(x'+a-\dfrac{a}{e}-ae+\dfrac{a}{e}\right)^2=\{ex'+a(e-1)\}^2$,

$$\text{or, } y^2+\{x'-a(e-1)\}^2=\{ex'+a(e-1)\}^2,$$
$$\text{or, } y^2+x'^2-2a(e-1)x'=e^2x'^2+2ae(e-1)x',$$
$$\text{or, } y^2+x'^2-2a(e+1)(e-1)x'=e^2x'^2,$$
$$\therefore\ y^2-x'^2(e^2-1)=2ax'(e^2-1),$$
$$\text{that is, } y^2=(e^2-1)(2ax'+x'^2).$$

Now, as A is to be accounted the origin, we may remove the accent, and the equation will be

$$y^2=(e^2-1)(2ax+x^2).$$

57. Suppose the origin to be at C.

Here, again, we might put x' for CM, and substitute $x'-\frac{a}{e}$ for x in the original equation.

Or, we may at once let x and y be the coordinates; then we shall have

$$\text{HM}=\text{CM}-\text{CH}=x-ae;$$

$$\text{DM}=\text{CM}-\text{CD}=x-\frac{a}{e};$$

$$\text{or } e\cdot\text{DM}=ex-a;$$

hence, equivalent to $\text{PM}^2+\text{HM}^2=e^2\text{DM}^2$, we have

$$y^2+(x-ae)^2=(ex-a)^2,$$

$$\text{or, } y^2=(e^2-1)(x^2-a^2).$$

Now, if we suppose $x=0$, we get $y^2=a^2(1-e^2)$, a negative value, since e is greater than unity; and therefore, assigning to y an impossible value, that is, denoting that the curve does not cut the axis of y. As, however, there will be a point on the curve having for its ordinate $y'=a\sqrt{e^2-1}$, we may take $\text{BC}=\text{CB}'=a\sqrt{e^2-1}$, and call this magnitude b;

thus $$b^2=a^2(e^2-1), \text{ or } e^2-1=\frac{b^2}{a^2}.$$

The equation, therefore, may be written

$$y^2=\frac{b^2}{a^2}(x^2-a^2),$$

which is reducible to the more symmetrical form

$$a^2y^2-b^2x^2=-a^2b^2, \quad \text{or } \frac{x^2}{a^2}-\frac{y^2}{b^2}=1;$$

which is the equation to the hyperbola, referred to the centre as origin, and is the one most frequently employed.

58. The above equation is convertible into the forms

$$y = \pm \frac{b}{a} \sqrt{x^2 - a^2}, \qquad (1)$$

$$x = \pm \frac{a}{b} \sqrt{y^2 + b^2}; \qquad (2)$$

which enable us readily to determine the form of the hyberbola.

If $y=0$, then (2) gives $x=\pm a$; therefore the hyperbola cuts the axis of x at the points A, A′.

If x be supposed a less magnitude than $+a$ or $-a$, then in (1) we shall have x^2-a^2 negative, and the values of y are imaginary. Therefore, no point of the hyperbola is situated between A and A′.

If x be supposed a greater magnitude than $+a$ or $-a$, then (1) shews that for every value of x there are two values of y, numerically equal, but contrary in sign. This implies that, for every point above the axis of x, there is a corresponding point at the same distance below the axis of x, so that the axis of x divides the curve symmetrically.

Moreover, as x increases, the values of y increase; and when x becomes indefinitely great, so also does y.

Accordingly, the hyperbola consists of two opposite and similar branches, respectively extending to the right of A and to the left of A′ indefinitely.

It may be here stated that the term axes, in reference to the hyperbola, is generally used to denote the lines AA′ and BB′, and that the former is called the transverse axis, the latter the conjugate axis.

It may be also mentioned that the ratio denoted by the symbol e is called the eccentricity of the hyperbola.

59. The hyperbola being symmetrically situated with respect to the axes AA′, BB′, there must be two foci, each having its corresponding directrix; and if A′S=AH, and A′E=DA, the curve can be described by means of S and LL′, as focus and directrix. (*See* fig. in **60.**)

60. The readiest mechanical method of describing an hyperbola is furnished by the following property:

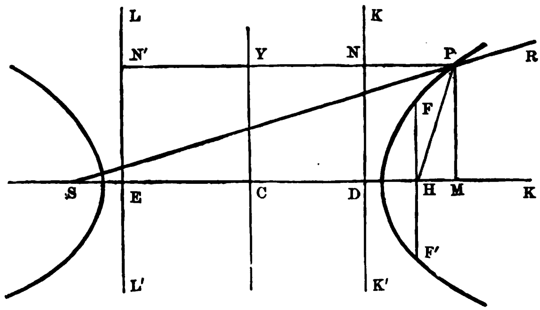
The difference of the focal distances of any point on the hyperbola is equal to the transverse axis.

For $SP = e \cdot PN' = e(CM + CE) = e\left(x' + \frac{a}{e}\right) = ex' + a,$

$$HP = e \cdot PN = e(CM - CD) = e\left(x' - \frac{a}{e}\right) = ex' - a,$$

$$\therefore SP - PH = 2a.$$

Accordingly, an hyperbola may be constructed as follows:



Let a ruler SR revolve round S in the plane of the paper, and let a string HPR, fastened to R and H, be of such length, that $SR - HPR = 2a$; then with a pencil, P, keep the string stretched against SR as the ruler revolves, and the point will trace out a portion of the hyperbola of which S and H are the foci, since SP—PH will be constantly $= 2a$.

61. The equation $y^2 = \frac{b^2}{a^2}(x^2 - a^2)$ may be put in the form

$$y^2 = \frac{b^2}{a^2}(x+a)(x-a);$$

hence, if PM be any ordinate, and C the origin,

$$PM^2 = \frac{BC^2}{CA^2} \cdot A'M \cdot MA,$$

$$\text{or } PM^2 : A'M \cdot MA :: BC^2 : CA^2.$$

62. The double ordinate FHF′, passing through the focus, is called the *latus rectum*. To find its length, we have $x = \text{CH} = ae$, which substituted for x in the equation $y^2 = \frac{b^2}{a^2}(x^2 - a^2)$, gives

$$y^2 = \frac{b^2}{a^2}(a^2e^2 - a^2) = b^2(e^2 - 1) = b^2 \cdot \frac{b^2}{a^2} = \frac{b^4}{a^2};$$

$$\text{hence } y = \text{FH} = \frac{b^2}{a}, \text{ and FHF}' = \frac{2b^2}{a}.$$

Tangent and Normal to an Hyperbola.

63. If we compare the equations to the ellipse and hyperbola,

$$\text{that for the ellipse being } y^2 = \frac{b^2}{a^2}(a^2 - x^2),$$

$$\text{that for the hyberbola } y^2 = \frac{b^2}{a^2}(x^2 - a^2)$$

$$= \frac{-b^2}{a^2}(a^2 - x^2),$$

it will appear that the latter may be derived from the former by writing $-b^2$ for b^2. Hence

To find the equation to the tangent at a given point (x', y') on an hyperbola:

Instead of repeating the procedure in art. **45**, we need only write $-b^2$ for b^2 in the equation there found, and we have

$$y - y' = \frac{b^2x'}{a^2y'}(x - x'),$$

$$\text{or } a^2yy' - b^2xx' = -a^2b^2.$$

64. *To express the tangent to an hyperbola in terms of the tangent of the angle which the line makes with the axis of x.*

Here the required expression may be at once derived from that in art. **46**; thus,

$$y = mx + \sqrt{a^2m^2 - b^2};$$

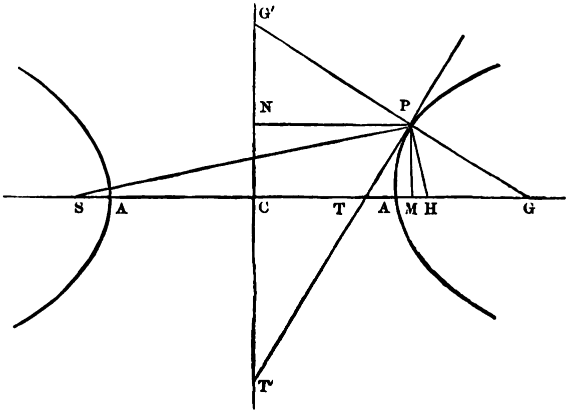
the double sign implying that *two* tangents may be drawn, making equal angles with the axis of x.

65. *To find the equation to the normal at a given point* (x', y') *on an hyperbola, and its intercepts on the axes.*

Here, again, replacing b^2 by $-b^2$, we obtain from art. **47**, for the equation to the normal PG,

$$y-y'=-\frac{a^2y'}{b^2x'}(x-x').$$

Also, for the intercepts, we should find, as in art. **47**,



$CG=e^2CM$, and $CG'=\frac{a^2e^2}{b^2}PM$, which are the lengths of the intercepts made by the normal PG with the axes.

Note. In determining the length of PG itself, we shall find

$$PM^2=(x'^2-a^2)(e^2-1),$$
$$GM=x'(e^2-1);$$

$$\therefore PG^2=PM^2+GM^2=(e^2-1)(1+e^2-1)x'^2-(e^2-1)a^2,$$
$$=(e^2-1)(e^2x'^2-a^2)=(e^2-1)(ex'+a)(ex'-a),$$

or $PG^2=(e^2-1)SP\cdot HP.$

66. *The focal distances of any point on the hyperbola make equal angles with the tangent at that point.*

This may be proved by the method employed in art. **48.** And hence also the angle between the focal distances of any point on the hyperbola is bisected by the normal at that point; or SPT=HPT.

67. *To find the locus of the intersection of a tangent to the hyperbola with the perpendicular drawn to it from either focus.*

By pursuing the method employed in art. **50**, the required locus will be ascertained to be a circle of which the transverse axis is a diameter; and the result will, as in that article, suggest a geometrical method of drawing a tangent to an hyperbola from any external point.

68. *To determine the perpendicular, p, from the centre of an hyperbola as origin, to the tangent at any point on the curve, in terms of the angle θ, which p makes with the axis of x.*

In the expression found in art. **51**, substitute $-b^2$ for b^2, and the equation determining p will be

$$p^2=a^2\cos^2\theta-b^2\sin^2\theta,$$

$$\text{or } p^2=a^2\cos^2\theta-a^2(e^2-1)\sin^2\theta,$$

$$=a^2(\cos^2\theta+\sin^2\theta)-a^2e^2\sin^2\theta,$$

$$\text{that is, } p^2=a^2(1-e^2\sin^2\theta)\,;$$

which is the same result as that for the ellipse.

69. All the properties of the hyperbola shewn in the preceding articles on that curve have been found similar to those of the ellipse. There are peculiar properties of the hyperbola, of which, in this introductory treatise, we do not consider the investigation to be necessary. We will, however, direct the attention of the student to one remarkable property of the hyperbola,—that of the asymptotes.

The equation to the hyperbola, viz.,

$$y^2=\frac{b^2}{a^2}(x^2-a^2),$$

may be put in the form

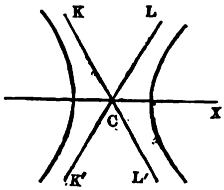
$$y^2=\frac{b^2x^2}{a^2}\left(1-\frac{a^2}{x^2}\right);$$

$$\text{hence } y=\pm\frac{bx}{a}\sqrt{\left(1-\frac{a^2}{x^2}\right)}.$$

Now, as x increases, the value of $\frac{a^2}{x^2}$ diminishes, so that that value becomes zero when, but not until, x becomes infinite; and then

$$y=\pm\frac{b}{a}x.$$

If, therefore, through the centre C, as origin, two straight lines, CL, CK, be drawn, making with the axis of x angles LCX, KCX, whose tangents are, respectively, $\frac{b}{a}$ and $-\frac{b}{a}$, these lines will continually approach the curve, and yet never meet it. For the equations to the lines are



$$y=\frac{b}{a}x, \text{ and } y=-\frac{b}{a}x,$$

and we have seen that y cannot become $=\pm\frac{b}{a}x$ till x becomes infinite.

The lines CL, CK are called *Asymptotes* to the hyperbola: the word asymptote, of Greek origin, signifying *without coincidence* or contact.

70. When the angle LCX is half a right angle, the tangent $\frac{b}{a}=1$, or $b=a$; and the value $b^2=a^2(e^2-1)$ gives $e^2=2$. The curve is then called an *equilateral* hyperbola, because

of the equality of the axes, and also a *rectangular* hyperbola, because of LCL′, the angle between the asymptotes, being a right angle.

71. *To find the equation to the rectangular hyperbola referred to the asymptotes as axes.*

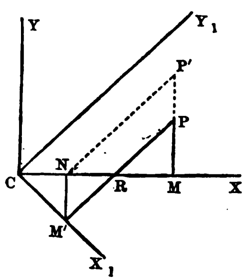
In the equation to the hyperbola,

$$a^2y^2-b^2x^2=-a^2b^2,$$

let $b=a$; then

$$x^2-y^2=a^2.$$

Now, let P be a point on the rectangular hyperbola, CX, CY the original axes, CX_1, CY_1 the axes coincident with the asymptotes.



Let $CM=x$, $PM=y$, $CM'=RM'=x'$, $PM'=y'$.

$$\begin{aligned} x=CM&=CN+NM \\ &=CM'\frac{CN}{CM'}+P'N\frac{NM}{P'N} \\ &=x'\cos 45°+y'\cos 45° \\ &=\tfrac{1}{2}\sqrt{2}(x'+y'); \end{aligned}$$

$$y=PM=PR\,\frac{PM}{PR}=(y'-x')\sin 45°=\tfrac{1}{2}\sqrt{2}(y'-x');$$

$$\therefore\ x^2-y^2=\tfrac{1}{2}(x'+y')^2-\tfrac{1}{2}(y'-x')^2=a^2,$$

$$\text{that is,}\quad 4x'y'=2a^2,$$

or, suppressing the accents,

$$xy=\tfrac{1}{2}a^2,$$

the equation required.

72. We will now propose a few Exercises on the hyberbola without an introduction of worked Examples.

Exercises [E].

1. The eccentricity of an hyperbola is 2: Shew that the equation to the curve is $y=3(x^2-a^2)$.

2. Required the eccentricity and latus rectum of an hyperbola whose equation is $2y^2=3(x^2+2n^2)$.

3. Shew that the tangent at A, the vertex of an ellipse or of an hyperbola, is at right angles to the transverse axis.

4. The tangent to an hyperbola at P cuts the transverse axis at T, and the tangent at the vertex A intercepts CP at E: Shew that the straight line TE is parallel to the straight line AP.

5. What relation must subsist between the constants, in order that the line $\frac{y}{m}-\frac{x}{n}=1$ may be a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$?

6. If an ellipse and an hyperbola have the same foci, the locus of the intersection of tangents to them, at right angles to each other, is a circle.

7. Find the locus of the intersection of tangents drawn from the fixed points A and B to a circle touching the straight line AB at a fixed point.

8. If P be any point on an hyperbola, shew that

$$\tan \tfrac{1}{2}\,\mathrm{PSH} \tan \tfrac{1}{2}\,\mathrm{PHS}=\frac{e-1}{e+1}.$$

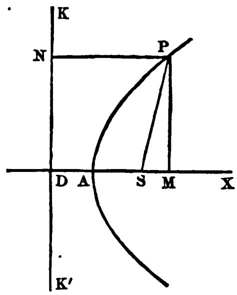
9. Through the extremities of PP′, a double ordinate of an ellipse, are drawn the straight lines A′PQ, P′AQ, meeting at Q: Shew that the locus of Q is an hyperbola having the same axes as the ellipse.

10. A circle is described, having the focus of an hyperbola as centre, and a diameter equal to the conjugate axis: Shew that the asymptotes are tangents to the circle at the points where they intersect the nearer directrix.

The Parabola.

73. The parabola is the locus of a point whose distance from a given fixed point is always equal to its distance from a given fixed straight line.

Let S be the given point, and KK′ the given line. Draw SD perpendicular to KK′. Let P be a point on the locus; join SP, and draw PM parallel to KK′, and PN parallel to DS.



Take D as the origin, DS as the axis of x, KK′ as the axis of y.

Bisect DS at A; suppose $AS=a$; and let x, y be the coordinates of P.

We have $SP=PN$; $\therefore SP^2=PN^2$; or $PM^2+SM^2=DM^2$;

that is, $y^2+(x-2a)^2=x^2$,

or, $y^2=4a(x-a)$;

which is the equation to the parabola, referred to the origin and axes assumed.

74. For the point where the parabola meets the axis of x, we have $y=0$, and the equation becomes $x=a$, signifying that the curve meets the axis at one point A, so that $DA=AS$.

A is called the vertex, AX the axis, S the focus, and KK′ the directrix, of the parabola.

75. Suppose the origin to be at A.

In this case, AM takes the place of DM; and if AM be called x', then $DM=x'+a$, that is, $x=x'+a$; whence, by substitution in the equation $y^2=4a(x-a)$, we have

$$y^2=4ax'.$$

Now, as A is to be accounted the origin, we may remove the accent, and we shall have

$$y^2=4ax,$$

which is the equation to the parabola referred to the vertex as origin, and is the one most frequently employed.

76. The above equation is convertible into the form

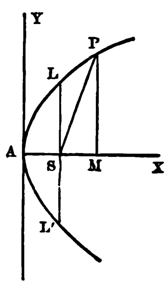
$$y=\pm 2\sqrt{ax},$$

which enables us readily to determine the form of the parabola.

If x be supposed negative, the values of y are imaginary. Therefore no point on the parabola is situated on the left of the origin.

For every positive value of x there are two values of y, numerically equal, but contrary in sign. This implies that for every point above the axis of x there is a corresponding point at the same distance below the axis of x; so that the axis of x divides the curve symmetrically.

Moreover, as x increases, the values of y increase, and when x becomes indefinitely great, so also does y.



Accordingly, the parabola is a non-central curve, extending indefinitely to the right of A.

77. The focal distance of any point P on the parabola, being equal to its distance from the directrix, we have

$$SP=AM+AS, \quad \text{or } SP=x+a.$$

The double ordinate LSL′ passing through the focus is called the *latus rectum*. To find its length, we have $x=AS=a$; $y^2=LS^2=4ax=4a^2$; hence $y=\pm 2a$; $\therefore$ LSL′$=4a$.

Tangent and Normal to a Parabola.

78. The tangent to a parabola may be regarded as a secant passing through two coincident points on the curve. (*See* art. **27.**)

To find the equation to the tangent at a given point on a parabola.

Let x', y' be the coordinates of P the given point, x'', y'' those of another point P′ on the curve, near the given point.

The equation to the straight line PP′ is

$$y-y'=\frac{y''-y'}{x''-x'}(x-x'). \qquad (1)$$

The equation to the parabola referred to the vertex as origin is

$$y^2=4ax.$$

Accordingly, since (x', y') and (x'', y'') are points on the curve,

$$y'^2=4ax', \text{ and } y''^2=4ax'';$$
$$\therefore\ y''^2-y'^2=4a(x''-x'),$$
$$\text{or } (y''+y')(y''-y')=4a(x''-x');$$
$$\therefore\ \frac{y''-y'}{x''-x'}=\frac{4a}{y''+y'};$$

hence the 1st equation, by substitution, will become

$$y-y'=\frac{4a}{y''+y'}(x-x'). \qquad (2)$$

Now, when the point (x'', y'') coincides with the point (x', y'), we have $x''=x'$, and $y''=y'$; in which case (2) becomes

$$y-y'=\frac{4a}{2y'}(x-x'),$$
$$\text{that is, } y-y'=\frac{2a}{y'}(x-x'),$$
$$\text{or } yy'-2ax+2ax'=y'^2=4ax';$$
$$\therefore\ yy'=2a(x+x').$$

79. *To express the tangent to a parabola in terms of the tangent of the angle which the line makes with the axis of x.*

We have seen that the tangent of the angle which the line makes with the axis of x is $\frac{2a}{y'}$.

From the equation $yy' = 2a(x + x')$

$$\text{we have} \quad y = \frac{2a}{y'}x + \frac{2ax'}{y'},$$

$$\text{or } y = \frac{2a}{y'}x + \frac{4ax'}{2y'},$$

$$\text{that is, } y = \frac{2a}{y'}x + \frac{y'^2}{2y'},$$

$$\text{or } y = \frac{2a}{y'}x + \frac{y'}{2}.$$

Let $\frac{2a}{y'} = m$; $\therefore \frac{y'}{2a} = \frac{1}{m}$, or $\frac{y'}{2} = \frac{a}{m}$; hence by substitution

$$y = mx + \frac{a}{m};$$

which is the equation to the tangent in terms of the tangent of the angle which the line makes with the axis of the parabola.

80. *To find the equation to the normal at any point on a parabola.*

Let (x', y') be the point of contact of a tangent to the parabola.

The equation to the tangent is

$$y = \frac{2a}{y'}x + \frac{y'}{2}.$$

The equation to the normal, as a straight line through (x', y'), is of the form

$$y - y' = m'(x - x');$$

and the condition of its being perpendicular to the tangent requires that $m' = -\frac{1}{m} = -\frac{y'}{2a}$; hence the equation to the normal at (x', y') is

$$y - y' = -\frac{y'}{2a}(x - x').$$

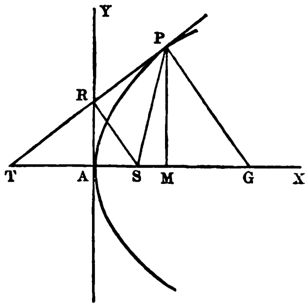
At G, where the normal PG meets the axis, $y=0$, and the above equation gives

$$-y'=-\frac{y'}{2a}(x-x'),$$

or $$1=\frac{1}{2a}(x-x'),$$

or $x-x'=2a$,

that is, $AG-AM=MG=2a$.



At T, where the tangent TP meets the axis, $y=0$, and the equation to the tangent, viz., $yy'=2a(x+x')$, becomes $x=-x'$, that is, $AM=AT$.

Hence also $$ST=x+a=SP.$$

Therefore the axis of a parabola, and the focal distance of any point on the curve to which a tangent is drawn, make equal angles with the tangent, and also equal angles with the normal; consequently $SG=SP$.

81. *To find the locus of the intersection of a tangent to the parabola with the perpendicular drawn to it from the focus.*

The equation to TP is

$$y=mx+\frac{a}{m}. \tag{1}$$

The equation to SR drawn from the focus whose coordinates are a, 0, is of the form

$$y=m'(x-a),$$

and that this line may be perpendicular to the former we put $m'=-\frac{1}{m}$, or

$$y=-\frac{1}{m}(x-a); \tag{2}$$

$$\text{hence } mx+\frac{a}{m}=-\frac{1}{m}(x-a),$$

$$\text{or } m^2x+a=a-x,$$
$$\text{or } (1+m^2)x=0;$$

but $1+m^2$ is not $=0$; $\therefore x=0$;

or, the locus of the intersection of a tangent to the parabola with the perpendicular on it from the focus is the axis of y.

82. We shall here propose a few Exercises on the parabola.

EXERCISES [F].

1. A straight line AL is drawn from the vertex of a parabola to an extremity of the latus rectum: Find the angle which AL makes with a tangent at L.

2. Find the length of TP, a tangent to a parabola at the point P, inclined to the axis at an angle PTX=60°.

3. LL′ being the latus rectum of a parabola whose vertex is A: Find the diameter of a circle described about the triangle LAL′.

4. If the tangent at a point P on a parabola be inclined to the axis at an angle of 30°, shew that the focal distance is equal to the latus rectum.

5. A tangent is drawn to a parabola at L, an extremity of the latus rectum: Find the length of the normal chord LGQ′.

6. AB is a fixed diameter of a circle whose centre is C; and DE is any chord parallel to AB: Find the locus of the intersection of CD with a straight line AQ drawn to the middle point of DE.

7. PSP′ is any focal chord of a parabola, the straight line AP cutting the latus rectum at Q: Shew that the straight line QP′ is bisected by the axis.

8. LS is the semi-latus rectum of a parabola APL, whose vertex is A and focus S; and D is the point where the axis meets the directrix; a straight line from D through P

meets the semi-latus rectum at N, and NQ is drawn parallel to DS, and SPQ is drawn to meet NQ: Shew that the locus of Q is a circle.

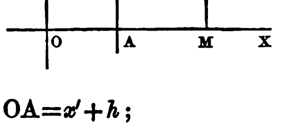
9. Two tangents are drawn to a parabola, and are inclined to the axis at angles the difference of whose cotangents is equal to a constant quantity d: Shew that the locus of the intersection of the tangents is a parabola equal to the original parabola.

Transformation of Coordinates.

83. It is often convenient to alter given coordinates with respect to origin, or direction, or both. This can be done with either oblique or rectangular axes. We confine our illustrations to the latter.

To transfer the origin of coordinates to another point, without changing the direction of the axes.

Let O′X′, O′Y′ be new axes parallel to the original axes OX, OY. Let the coordinates of the new origin referred to the old origin be $OA=h$, $O'A=k$. Let x, y be the original coordinates of a point P, and x', y' the new coordinates of the same point.

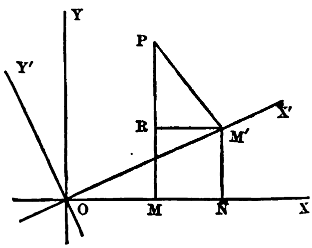


$$x=OM=AM+OA=x'+h;$$
$$y=PM=PM'+M'M=y'+k;$$

which are the values of x and y that must be substituted in order to express what the equation to any locus becomes when referred to the new axes.

84. *To change one system of rectangular axes to another with the same origin.*

Let OX′, OY′ be new rectangular axes, having the same origin as OX, OY, the original axes. Let x, y be the original coordinates of a point P, and x', y' the new coordinates of the point.



Draw PM, PM′ parallel to OY, OY′, and M′N, M′R parallel to OY, OX.

The axes have been turned round O, through an angle XOX′; let that angle $=\theta$;

then
$$x = \mathrm{OM} = \mathrm{ON} - \mathrm{RM}',$$
$$y = \mathrm{PM} = \mathrm{PR} + \mathrm{M'N};$$

but the angle $\mathrm{RPM}' = \mathrm{XOX}' = \theta$;

$$\therefore\ \mathrm{RM}' = \mathrm{PM}' \sin\theta; \qquad \text{and}\ \ \mathrm{PR} = \mathrm{PM}' \cos\theta;$$
$$\mathrm{ON} = \mathrm{OM}' \cos\theta; \qquad \text{and}\ \ \mathrm{M'N} = \mathrm{OM}' \sin\theta.$$
$$\therefore\ x = x' \cos\theta - y' \sin\theta;$$
$$y = x' \sin\theta + y' \cos\theta;$$

which are the values of x and y that must be substituted in order to express what the equation to any locus becomes when referred to the new axes.

To make this case include a transfer of the origin to the point (h, k), we have only to substitute $x''+h$ for x', and $y''+k$ for y'.

85. The general equation of the second degree is

$$\mathrm{A}x^2 + \mathrm{B}xy + \mathrm{C}y^2 + \mathrm{D}x + \mathrm{E}y + \mathrm{F} = 0;$$

and it most frequently represents some curve.

It *may*, however, represent two straight lines. Thus, if we have

$$6x^2 - 11xy + 3y^2 - 3x + 8y - 3 = 0,$$

then solving this as a quadratic for y, we shall obtain as the two values of y,

$$y = 3x - 3, \text{ and } = \tfrac{2}{3}x + \tfrac{1}{3},$$

shewing that the given equation is the product of

$$y-3x+3=0$$

$$\text{and } 3y-2x-1=0,$$

and therefore represents two straight lines.

We proceed to shew how transformation of coordinates may be applied for the reduction of the general equation of the second degree to the forms by which loci are readily distinguished.

86. Let us begin by examining how the terms involving the simple powers of x and y may be removed.

If we transfer the origin to a point (h, k), without altering the direction of the rectangular axes; that is, if we substitute $x'+h$ for x, and $y'+k$ for y, we shall have

$$\begin{aligned} &Ax'^2+2Ahx'+Ah^2+Bx'y'+Bkx'+Bhy'+Bhk \\ &+Cy'^2+2Cky'+Ck^2+Dx'+Dh+Ey'+Ek+F=0\,; \end{aligned}$$

that is,

$$\begin{aligned} &Ax'^2+Bx'y'+Cy'^2+(2Ah+Bk+D)x'+(Bh+2Ck+E)y' \\ &+Ah^2+Bhk+Ck^2+Dh+Ek+F=0. \end{aligned}$$

Now, to try to get rid of the terms of x' and y', let us equate their coefficients to zero; thus,

$$2Ah+Bk+D=0\,; \qquad \text{and } Bh+2Ck+E=0\,;$$

$$\therefore\ k=\frac{-2Ah-D}{B}=\frac{-Bh-E}{2C}\,; \qquad (1)$$

$$h=\frac{-Bk-D}{2A}=\frac{-2Ck-E}{B}. \qquad (2)$$

From (1) we have $\quad h=\dfrac{2C\cdot D-B\cdot E}{B^2-4A\cdot C}.$

From (2) we have $\quad k=\dfrac{2A\cdot E-B\cdot D}{B^2-4A\cdot C}.$

It appears then that such values of h and k can be assigned as will make the coefficients of x and y = zero, except when $B^2-4A\cdot C=0$.

87. When $B^2-4A\cdot C=0$, or $B^2=4A\cdot C$, we do not get rid of the simple powers of x and y by equating their coefficients to zero as above; but we can get rid of the product of the variables by putting

$$x=x' \cos \theta - y' \sin \theta,$$
$$y=x' \sin \theta + y' \cos \theta.$$

The general equation then becomes

$$A(x'^2 \cos^2\theta - 2x'y' \cos\theta \sin\theta + y'^2 \sin^2\theta) + B(x'^2 \sin\theta \cos\theta - x'y' \sin^2\theta + x'y' \cos^2\theta - y'^2 \sin\theta \cos\theta) + C(x'^2 \sin^2\theta + 2x'y' \sin\theta \cos\theta + y'^2 \cos^2\theta) + D(x' \cos\theta - y' \sin\theta) + E(x' \sin\theta + y' \cos\theta) + F = 0;$$

or, since $2 \sin\theta \cos\theta = \sin 2\theta$, and $\cos^2\theta - \sin^2\theta = \cos 2\theta$,

$$\left.\begin{array}{l} x'^2(A\cos^2\theta + \tfrac{1}{2}B \sin 2\theta + C \sin^2\theta) \\ +x'y'\{(C-A)\sin 2\theta + B \cos 2\theta\} \\ +y'^2(A \sin^2\theta - \tfrac{1}{2}B \sin 2\theta + C \cos^2\theta) \\ +x'(D\cos\theta + E \sin\theta) \\ +y'(E\cos\theta - D\sin\theta) \\ +F \end{array}\right\} = 0.$$

Now, the tangent of an angle may be of any magnitude; and therefore we may put

$$\frac{B}{A-C} = \tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta},$$

whereby we make $\quad -B \cos 2\theta = (C-A) \sin 2\theta,$

and thus, by substitution in the coefficient of $x'y'$, that coefficient will become zero.

We can, therefore, always get rid of the product of the variables by turning the axes through an angle θ such that

$$\tan 2\theta = \frac{B}{A-C}.$$

Let us now seek the corresponding values of the coefficients of x'^2 and y'^2.

$$2A \cos^2\theta + 2C \sin^2\theta = A(2 - 2\sin^2\theta) + 2C \sin^2\theta;$$

but, by Trigonometry, $2\sin^2\theta=1-\cos 2\theta$; thus making the expression become

$$=A(1+\cos 2\theta)+C(1-\cos 2\theta)$$
$$=A+C+(A-C)\cos 2\theta;$$
$$\therefore A\cos^2\theta+C\sin^2\theta=\tfrac{1}{2}\{A+C+(A-C)\cos 2\theta\}.$$

Similarly,

$$A\sin^2\theta+C\cos^2\theta=\tfrac{1}{2}\{A+C-(A-C)\cos 2\theta\}.$$

Hence the terms involving x'^2 and y'^2 become

$$x'^2\cdot\tfrac{1}{2}\{A+C+(A-C)\cos 2\theta+B\sin 2\theta\}$$
$$+y'^2\cdot\tfrac{1}{2}\{A+C-(A-C)\cos 2\theta-B\sin 2\theta\}.$$

$$\text{Now, } \tan^2 2\theta=\frac{\sin^2 2\theta}{\cos^2 2\theta}=\frac{B^2}{(A-C)^2};$$
$$\therefore B\sin 2\theta=\frac{B^2}{\sqrt{(A-C)^2+B^2}},$$
$$(A-C)\cos 2\theta=\frac{(A-C)^2}{\sqrt{(A-C)^2+B^2}};$$

so that the terms involving x'^2 and y'^2 become

$$x'^2\cdot\tfrac{1}{2}\{A+C+\sqrt{(A-C)^2+B^2}\}$$
$$+y'^2\cdot\tfrac{1}{2}\{A+C-\sqrt{(A-C)^2+B^2}\};$$

where it will be observed that the product of the coefficients is $\frac{1}{4}(4A\cdot C-B^2)$; if, therefore, $4A\cdot C-B^2=0$, one of these coefficients must vanish.

In this case, if the coefficient of $x'^2=0$, then the general equation, having lost x'^2 and $x'y'$, becomes

$$C'y'^2+D'x'+E'y'+F=0,$$

which may be written

$$y'^2+\frac{E'}{C'}y'+\frac{D'}{C'}x'+\frac{F}{C'}=0,$$

$$\text{or, } \left(y'+\frac{E'}{2C'}\right)^2=-\frac{D'}{C'}x'-\frac{F}{C'}+\frac{E'^2}{4C'},$$

$$\text{or, } \left(y'+\frac{E'}{2C'}\right)^2=-\frac{D'}{C'}\left(x'+\frac{F}{D'}-\frac{E'^2}{4C'\cdot D'}\right).$$

Let $-\frac{E'}{2C'}$ be called P, and let $\frac{E'^2}{4C' \cdot D'}-\frac{F}{D'}$ be called R;

then, $$(y'-P)^2=-\frac{D'}{C'}(x'-R);$$

hence, if D be not $=0$, and the origin be transferred to the point (R, P),

writing $x''+R$ for x', and $y''+P$ for y', we shall have

$$y''^2=-\frac{D'}{C'}x'',$$

which is the equation to a parabola, the coordinates to the vertex being R and P.

88. We will now proceed to Examples and Exercises on the use of those expedients that we have found available for the transformation of coordinates.

EXAMPLES.

Ex. (1). Transform the equation $x+2\sqrt{xy}+y=c$, by turning the axes of the coordinates through half a right angle.

Here $\theta=45°$, and $\sin\theta=\cos\theta=\frac{1}{2}\sqrt{2}$.

Putting $x=x'\cos\theta-y'\sin\theta$, and $y=x'\sin\theta+y'\cos\theta$, we have $x=\frac{1}{2}\sqrt{2}(x'-y')$; and $y=\frac{1}{2}\sqrt{2}(x'+y')$;

$\therefore x+y+2\sqrt{xy}=c$ becomes

$$\sqrt{2}x'+2\sqrt{\tfrac{1}{2}(x'^2-y'^2)}=c;$$
$$\text{or, } (\sqrt{2}x'-c)^2=2x'^2-2y'^2;$$
$$\text{or, } y'^2=c\sqrt{2}x'-\tfrac{1}{2}c^2.$$

Ex. (2). Transform the equation

$$5x^2-4xy+2y^2-3x=0,$$

by removing the simple power of x and the product of the variables.

Here $A=5$, $B=-4$, $C=2$, $D=-3$, $E=0$, $F=0$; and $B^2-4A \cdot C$ being $=16-40=-24$, we can get rid of the

simple power of x by transferring the origin to the point (h, k), taking

$$h=\frac{2C\cdot D-B\cdot E}{B^2-4A\cdot C}=\frac{-12}{-24}=\frac{1}{2},$$

$$k=\frac{2A\cdot E-B\cdot D}{B^2-4A\cdot C}=\frac{-12}{-24}=\frac{1}{2}.$$

Thus for x take $x'+\frac{1}{2}$, and for y take $y'+\frac{1}{2}$; then,

$$5(x'+\tfrac{1}{2})^2-4(x'+\tfrac{1}{2})(y'+\tfrac{1}{2})+2(y'+\tfrac{1}{2})^2-3(x'+\tfrac{1}{2})=0;$$

$$\text{or } 20x'^2-16x'y'+8y'^2-3=0.$$

Now, to remove the product of the variables, take $\tan 2\theta$ $=\frac{B}{A-C}=\frac{-4}{5-2}=-\frac{4}{3}$.

By Trigonometry, $\tan\theta=\frac{\sqrt{(1+\tan^2 2\theta)}-1}{\tan 2\theta}=\frac{-5-3}{-4}=2$;

$$\therefore \sin\theta=\tfrac{2}{5}\sqrt{5}, \text{ and } \cos\theta=\tfrac{1}{5}\sqrt{5}.$$

Hence, putting $x''\cos\theta-y''\sin\theta$ for x',

$x''\sin\theta+y''\cos\theta$ for y',

we have

$$\begin{aligned}
20(\tfrac{1}{5}\sqrt{5}x''-\tfrac{2}{5}\sqrt{5}y'')^2 &= 4x''^2-16x''y''+16y''^2\\
8(\tfrac{2}{5}\sqrt{5}x''+\tfrac{1}{5}\sqrt{5}y'')^2 &= 6\tfrac{2}{5}x''^2+6\tfrac{2}{5}x''y''+1\tfrac{3}{5}y''^2\\
-16(\tfrac{1}{5}\sqrt{5}x''-\tfrac{2}{5}\sqrt{5}y'')(\tfrac{2}{5}\sqrt{5}x''+\tfrac{1}{5}\sqrt{5}y'') &= -6\tfrac{2}{5}x''^2+9\tfrac{3}{5}x''y''+6\tfrac{2}{5}y''^2\\
\text{sum}\qquad & \overline{4x''^2+24y''^2=3;}
\end{aligned}$$

or, suppressing the accents,

$$\frac{x^2}{\frac{3}{4}}+\frac{y^2}{\frac{1}{8}}=1,$$

which is the equation to an ellipse whose axes are $2a=\sqrt{3}$, and $2b=\frac{1}{2}\sqrt{2}$.

Ex. (3). Required the latus rectum, and the coordinates of the vertex, of the parabola whose equation is

$$x^2-4xy+4y^2-4x-2y+10=0.$$

Here A=1, B=−4, C=4, D=−4, E=−2, F=10.

B^2-4AC being $=0$, we cannot remove the origin. Let us, however, turn the axes through an angle θ such that $\tan 2\theta=\frac{B}{A-C}=\frac{4}{3}$.

We have $\tan\theta=\frac{1}{2}$, $\therefore$ $\sin\theta=\frac{1}{5}\sqrt{5}$, and $\cos\theta=\frac{2}{5}\sqrt{5}$.

Hence, putting $\frac{2}{5}\sqrt{5}x'-\frac{1}{5}\sqrt{5}y'$ for x,

$$\tfrac{1}{5}\sqrt{5}x'+\tfrac{2}{5}\sqrt{5}y' \text{ for } y,$$

the given equation is reduced to

$$5y'^2-2\sqrt{5}x'+10=0,$$

where $C'=5$, $D'=-2\sqrt{5}$, $E'=0$, $F=10$;
hence the formula

$$\left(y'+\frac{E'}{2C'}\right)^2=-\frac{D'}{C'}\left(x'+\frac{F}{D'}-\frac{E'^2}{4C'\cdot D'}\right)$$

gives
$$y'^2=\tfrac{2}{5}\sqrt{5}(x'-\sqrt{5}),$$
showing the latus rectum $4a=\frac{2}{5}\sqrt{5}$, and the coordinates of the vertex $-\sqrt{5}$ and 0.

Ex. (4). Trace the curve $y=x-x^2$.

We have here $x^2+y-x=0$. Substitute y' for x and x' for y; then

$$y'^2+x'-y'=0,$$

where $C'=1$, $D'=1$, $E'=-1$, $F=0$;
hence the formula in the preceding example gives

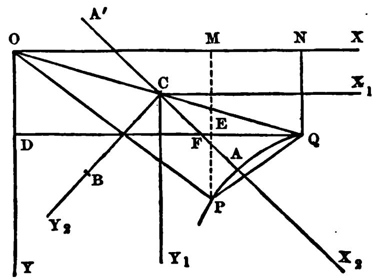
$$(y'-\tfrac{1}{2})^2=-(x'-\tfrac{1}{4});$$

now replacing y' by x and x' by y, we have

$$(x-\tfrac{1}{2})^2=-(y-\tfrac{1}{4}),$$

which is the equation to a parabola, having its axis parallel to that of y, the coordinates of its vertex $\frac{1}{2}$ and $\frac{1}{4}$, and its latus rectum 1.

Ex. (5). $DQ=l$ is a horizontal line, and $DO=p$ a perpen-



dicular erected upon it. A cord OPQ, fixed at O, is drawn over the point Q, while at P a weight, freely suspended, slides along the cord. Required the locus of P.

Let the coordinates of Q be $ON=l$, $QN=p$; and let those of P be $OM=x$, $PM=y$.

The triangles OPM, EPQ being similar, $\frac{EQ}{OM}=\frac{PE}{PM}$,

that is, $$\frac{l-x}{x}=\frac{y-p}{y},$$

$$\text{or } 2xy-px-ly=0. \qquad (1)$$

Here $A=0$, $B=2$, $C=0$, $D=-p$, $E=-l$; hence $h=\frac{1}{2}l$, and $k=\frac{1}{2}p$; and if we put $x'+\frac{1}{2}l$ for x, and $y'+\frac{1}{2}p$ for y, we have, by substitution in (1),

$$x'y'=\tfrac{1}{4}lp,$$

which is the equation to an hyperbola whose asymptotes are the axes of x and y, and in the present instance rectangular.

Accordingly, since $\frac{1}{4}lp=\frac{1}{4}ON\cdot QN$, the locus of P is an equilateral hyperbola whose centre is C the middle point of the diagonal OQ, and its transverse axis A′A making with DQ an angle QFA$=45°$.

Exercises [G].

1. Shew that the equation $3xy-4x=a^2$ will become $x^2-9y^2=2a^2$, by turning the axes through an angle whose tangent is 3.

2. The equation to a rectangular hyperbola being $x^2-y^2=a^2$; transform that equation by turning the axes in the negative direction through half a right angle.

3. Find the locus represented by the equation
$$4x^2+12xy+9y^2-1=0.$$

4. Find the axes of the curve $x^2+xy+y^2=1$.

5. What is represented by the equation
$$9x^2-30xy+25y^2+21x-35y+10=0\,?$$

6. Determine the axes of the curve which is the locus of the equation

$$x^2-6xy+y^2-6x+2y+5=0.$$

7. What is represented by the equation

$$y^2-2xy-x^2+2=0\,?$$

8. Determine the nature and position of the curve represented by the equation

$$x^2-2xy+y^2-8x+16=0.$$

9. What is the geometrical construction of the following equation:

$$y^2-2xy+x^2-2y-2x-3=0\,?$$

10. Find the axes and position of the curve represented by the equation

$$3x^2-2xy+y^2-4x+2y-3=0.$$

11. The base of a triangle is $=2n$, and the difference of the angles at the base is 30°. Find the locus of the vertex.

12. Shew that the lines drawn from the angles of a triangle perpendicular to the opposite sides meet at a point; and that the locus of that point, for all equal triangles on the same base, is a parabola whose latus rectum is the altitude of the triangle.

Answers to the Exercises.

[B].

1. $7x+8y=19$. 2. $19x-11y=132$.
3. $3x=7y$; and $3x-7y=11$. 4. $3x-5y=20$.
5. $109y+(448\pm305\sqrt{3})x-5(1102\pm305\sqrt{3})=0$.
6. 20. 7. 35. 9. $\frac{4}{3}$. 10. $4x+5y=0$. 11. $1383\frac{1}{2}$.
12. 383. 13. Taking (x', y') for P, and (x'', y'') for Q, area$=\frac{1}{2}(x''y'-x'y'')$; OP, OQ, PQ, $=13, 15, \sqrt{58}$.
14. $(2, 3), (5, 2), (-8, 28)$; area$=32\frac{1}{2}$. 15. $y=\frac{2}{7}x+\frac{69}{14}$; DE is parallel to AC. 16. $(a'-a)y-(b'-b)x-a'b+ab'=0$; and $(a'-a)y+(b'-b)x-a'b'+ab=0$.
17. The eqn. to OR is $x=0$; QR, $4y-3x=56$;
PR, $5x+12y=168$; area of paral^m$=168$.
18. $y=-3x+9$; $3y=x+7$; $y=-3x+14$; $3y=x+12$.
19. $391x+187y=2184$. 21. tan CAB$=m'$; tan. CBA

$$=-m;\ \tan\ \mathrm{ACB}=\frac{m-m'}{1+mm'};\ \mathrm{AB}=\frac{c}{m};$$

$$\mathrm{AC}=\frac{c}{m-m'}\sqrt{(1+m'^2)};\ \mathrm{BC}=\frac{cm'}{m(m-m')}\sqrt{(1+m^2)}.$$

[C].

1. The radius and the coordinates of the centre are, (i.) 7; 3 and 4; (ii.) 3; -1 and -2; (iii.) $4\sqrt{5}$; 3 and 9; (iv.) $\sqrt{5}$; $-\frac{1}{2}$ and 1.
2. (i.) $(5, 3)$ and $(-3, -5)$; (ii.) $(2\frac{1}{5}, -5\frac{2}{5})$ and $(-3, 5)$.
3. (i.) The line is a tangent to the circle at the point $(\frac{3}{5}, -\frac{4}{5})$; (ii.) The line meets the circle at the point $(-1, 0)$, which is an extremity of the diameter in the axis of x, and at the point $(-\frac{9}{41}, -\frac{40}{41})$.

4. (i.) Origin on the circumf.; points of intersection (24, 10), which is an extremity of the diam. through the origin, and (3·84, −5·12). (ii.) Points of intersection (6, $-3\frac{1}{2}$), (4, −5).

5. $y-b=m(x-a)\pm c\sqrt{(1+m^2)}$. 6. 32.

7. $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{c^2}$.

8. $x=\frac{1}{2a}(a^2+c^2-c'^2)$; $y=\pm\frac{1}{2a}\sqrt{(a+c+c')(a+c-c')(a-c+c')(c-a+c')}$.

9. The centre is on the axis of x at $\frac{m^2+1}{m^2-1}a$ from the origin; the rad.$=\frac{2m}{m^2-1}a$; coordinates of points of intersection with the axis of x are $x=\frac{m+1}{m-1}a$ and $x=\frac{m-1}{m+1}a$.

11. $x^2+y^2=2\frac{1}{2}$. 12. $x^2+y^2-ax=c^2-\frac{1}{2}a^2$.

13. $c=\pm\frac{1}{2}\sqrt{2}(a-b)$. 14. $x^2+y^2-\frac{2c}{1+m^2}(x-my)=0$.

[D].

1. (i.) $\frac{1}{2}$; (ii.) $\frac{1}{5}\sqrt{10}$; (iii.) $\frac{12}{13}$. 2. $e=\frac{1}{2}\sqrt{2}$.

3. $1\frac{1}{4}$ and $\frac{3}{4}\sqrt{15}$. 4. $y+ae^2=\frac{x}{e}$. 5. 21 and 15.

10. $3y+\sqrt{5}x-9a=0$; the intercept on the axis of y is $3a$; that on the axis of x is $\frac{9a}{5}\sqrt{5}$.

[E].

2. $e=\frac{1}{2}\sqrt{10}$; lat. rect.$=3a$. 5. $\frac{a^2}{n^2}-\frac{b^2}{m^2}=1$.

7. An hyperbola, if N be between A and B, and be not the middle point of AB. If N be on AB produced, the locus is an ellipse.

[F].

1. An angle whose tangent is $\frac{1}{3}$. 2. $\frac{4}{3}a$. 3. $5a$.

5. $8a\sqrt{2}$. 6. A parabola of which a tangent to the circle at A is the directrix, and the centre of the circle is the focus.

[G].

2. $x'y'=\frac{1}{2}a^2$. 3. Two parallels, each inclined to the axis of x at an angle whose tangent $=-\frac{2}{3}$, and whose distance from each other $=\pm\frac{1}{10}\sqrt{13}$. 4. Axes of the ellipse, $2a=\frac{2}{3}\sqrt{6}$, $2b=2\sqrt{2}$. 5. Two parallels whose eqns. are $y=\frac{3}{5}x+\frac{2}{5}$ and $y=\frac{3}{5}x+1$. 6. Axes of the hyperbola, $2a=2\sqrt{2}$, and $2b=2$. 7. A rectangular hyperbola, each semiaxis $=2^{\frac{1}{4}}$. 8. A parabola; the coordinates $1\frac{1}{2}\sqrt{2}$ and $-\sqrt{2}$. 9. A parabola; latus rectum $=\sqrt{2}$. 10. An ellipse; the coordinates of the centre $\frac{1}{2}$ and $-\frac{1}{2}$, and the axes $1\frac{1}{2}\sqrt{2+\sqrt{2}}$ and $1\frac{1}{2}\sqrt{2-\sqrt{2}}$.

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